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On the moment map for the variety of Lie algebras[☆]

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Abstract

We consider the moment map $m: \mathbb{P}V_n \rightarrow \mathfrak{iu}(n)$ for the action of $GL(n)$ on $V_n = \Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$. The critical points of the functional $F_n = \|m\|^2: \mathbb{P}V_n \rightarrow \mathbb{R}$ are studied, in order to understand the stratification of $L_n \subset \mathbb{P}V_n$ defined by the negative gradient flow of F_n , where L_n is the algebraic subset of all Lie algebras. We obtain a description of the critical points which lie in L_n in terms of those which are nilpotent, as well as the minima and maxima of $F_n: L_n \rightarrow \mathbb{R}$. A characterization of the critical points modulo isomorphism, as the union of categorical quotients of suitable actions is considered, and some applications to the study of L_n are given.

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1. Introduction

The space of all complex Lie algebras of a given dimension n can be naturally identified with the set \mathcal{L}_n of all Lie brackets on \mathbb{C}^n . Since the Jacobi identity is determined by polynomial conditions, \mathcal{L}_n is an algebraic subset of the vector space V_n of skew-symmetric bilinear maps from $\mathbb{C}^n \times \mathbb{C}^n$ to \mathbb{C}^n . The isomorphism class of a

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Lie algebra $\mu \in \mathcal{L}_n$ is then given by the orbit $GL(n) \cdot \mu$ under the ‘change of basis’ action of $GL(n)$ on V_n . This action is very unpleasant from the point of view of invariant theory since any $\mu \in V_n$ is unstable (i.e. $0 \in \overline{GL(n) \cdot \mu}$), which makes very difficult the study of the quotient space $\mathcal{L}_n/GL(n)$ parameterizing Lie algebras up to isomorphism.

Nevertheless, Kirwan [8] and Ness [17] have showed that the moment map for an action can be used to study the orbit space of the null-cone (set of unstable vectors). Let $m: \mathbb{P}V \rightarrow \mathbb{R}$ be the moment map for the action of a complex reductive Lie group G with maximal compact subgroup K on a vector space V . An orbit $G \cdot v$ is closed if and only if $G \cdot [v]$ meets $m^{-1}(0)$ and the intersection is a single K -orbit (see [7]), where $[v]$ denotes the class of v in the projective space $\mathbb{P}V$. Let $X \subset \mathbb{P}V$ be a G -invariant projective algebraic variety. The so-called categorical quotient $X//G$ parameterizing closed orbits is homeomorphic to $(X \cap m^{-1}(0))/K$, which is precisely the symplectic reduction when X is nonsingular.

It is proved in [8,17] that the remaining critical points of $F = \|m\|^2: \mathbb{P}V \rightarrow \mathbb{R}$ (i.e. such that $F(x) > 0$) are all in the null-cone and in some sense, their orbits play the same role that the closed orbits in the set of semistable points (i.e. $0 \notin \overline{G \cdot v}$). For instance, if C denotes the set of critical points of F then $C \cap G \cdot [v]$ is either empty or a single K -orbit, and they are minima of $F|_{G \cdot [v]}$. It is then natural to consider a wider quotient $X///G$, which shall be called *Kirwan–Ness quotient*, given by

$$X///G = (X \cap C)/K.$$

Recall that if the action contains the homotheties $\{v \mapsto tv: t \in \mathbb{C}^*\}$, then we may also define the Kirwan–Ness quotient for the action of G on an algebraic G -variety $X \subset V$ by

$$X///G = \{v \in X: [v] \in C\}/\mathbb{C}^*K,$$

and clearly $X///G = \pi(X)///G$, where $\pi: V \setminus \{0\} \rightarrow \mathbb{P}V$ is the usual projection map. This new quotient $X///G$ is not a projective algebraic variety (not even Hausdorff) as in the case of $X//G$, but nevertheless, its topology is not so wild. Indeed, $X///G$ can be decomposed as a disjoint union of projective algebraic varieties with a respectable frontier property, coming from the stratification defined by the negative gradient flow of F (see Section 2).

Let us now go back to the variety of Lie algebras. We note that $\mathcal{L}_n/GL(n)$ consists of only one point, the abelian Lie algebra, and we will show that to consider $SL(n)$ -orbits does not help much, indeed, $\mathcal{L}_n//SL(n) =$ semisimple Lie algebras (and the abelian Lie algebra). The aim of this paper is to initiate the study of the Kirwan–Ness quotient $\mathcal{L}_n///GL(n)$, or $L_n///GL(n)$, where $L_n = \pi(\mathcal{L}_n)$. How special are the Lie algebras which are isomorphic to a critical point of $F_n: \mathbb{P}V_n \rightarrow \mathbb{R}$? The main result is that the study of $\mathcal{L}_n///GL(n)$ reduces essentially to the understanding of $\mathcal{N}_n///GL(n)$, where \mathcal{N}_n is the subvariety of nilpotent Lie algebras. Our real goal is, however, the possible applications of this ‘moment map’ approach to the study of degenerations, rigidity and of the irreducible components of \mathcal{L}_n .

The moment map is in our case a $U(n)$ -equivariant function $m: \mathbb{P}V_n \mapsto \mathfrak{iu}(n)$, where $\mathfrak{iu}(n)$ denotes the space of hermitian matrices. It comes from the Hamiltonian action of $U(n)$ on the symplectic manifold $\mathbb{P}V_n$. One fixes an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{C}^n and considers the corresponding $U(n)$ -invariant inner products on V_n and $\mathfrak{iu}(n)$, respectively, naturally associated with $\langle \cdot, \cdot \rangle$. For each $\mu \in V_n$ with $\|\mu\| = 1$, $m([\mu])$ is defined as the derivative at the identity of the function $GL(n) \mapsto \mathbb{R}, g \mapsto \|g \cdot \mu\|^2$. It will be proved that

$$m([\mu]) = M_\mu := -4 \sum_i (\text{ad}_\mu X_i)^* \text{ad}_\mu X_i + 2 \sum_i \text{ad}_\mu X_i (\text{ad}_\mu X_i)^*, \quad \|\mu\| = 1,$$

where $\{X_1, \dots, X_n\}$ is any orthonormal basis of \mathbb{C}^n . We now consider the functional $F_n: \mathbb{P}V_n \mapsto \mathbb{R}$ given by $F_n([\mu]) = \|m([\mu])\|^2 = \text{tr } M_\mu^2$. Some of the remarkable properties of F_n , its gradient flow and their critical points can be summarized as follows (see Section 2):

- $\text{grad}(F_n)_{[\mu]} = -8\pi_* \delta_\mu(M_\mu)$, $\|\mu\| = 1$, where $\delta_\mu: \mathfrak{gl}(n) \mapsto V_n$ coincides with the coboundary operator relative to adjoint cohomology when μ is a Lie algebra, and π_* denotes the derivative of $\pi: V_n \setminus \{0\} \mapsto \mathbb{P}V_n$.
- The gradient of F_n is then always tangent to the $GL(n)$ -orbits, and so $[\mu]$ is a critical point of F_n if and only if it is such for $F_n|_{GL(n) \cdot [\mu]}$. Thus, the negative gradient flow of F_n stays in the orbit of the starting point $[\lambda]$ and therefore it gives rise to a distinguished degeneration from λ to a critical point μ (i.e. $\mu \in \overline{GL(n) \cdot \lambda}$).
- If $[\mu]$ is a critical point of F_n then $F_n|_{GL(n) \cdot [\mu]}$ attains its minimum value at $[\mu]$, and any other critical point in $GL(n) \cdot [\mu]$ belongs to $U(n) \cdot [\mu]$.
- $[\mu] \in \mathbb{P}V_n$ is a critical point of F_n if and only if $M_\mu = cI + D$ for some $c \in \mathbb{R}$ and $D \in \text{Der}(\mu)$. In this case, a positive multiple of D has nonnegative integer eigenvalues k_i , say with multiplicities d_i , and c can be computed in terms of these integers. The critical point $[\mu]$ is said to be of *type* $\alpha = (k_1 < \dots < k_r; d_1, \dots, d_r)$.
- Any critical point $[\mu]$ admits a $\mathbb{Z}_{\geq 0}$ -gradation. If $\mu \in \mathcal{L}_n$ then $k_1 > 0$ if and only if μ is nilpotent, and so any nilpotent Lie algebra which is a critical point of F_n is \mathbb{N} -graded.
- There are finitely many types of critical points, say $\alpha_1, \dots, \alpha_s$.
- If C_α is the set of critical points of F_n of type α , then the quotient space $C_\alpha/U(n) = GL(n) \cdot C_\alpha/GL(n)$ is homeomorphic to the categorical quotient of a suitable action, and so it is a projective algebraic variety (see [16]).
- Let $S_\alpha \subset \mathbb{P}V_n$ be the set of all the points which are carried by the negative gradient flow of F_n to a critical point of type α . Then

$$\mathbb{P}V_n = S_{\alpha_1} \cup \dots \cup S_{\alpha_s}$$

determines a stratification of $\mathbb{P}V_n$, for which each stratum S_{α_i} is locally closed, irreducible and nonsingular.

We are interested in the stratification of L_n given by

$$L_n = (S_{\alpha_1} \cap L_n) \cup \cdots \cup (S_{\alpha_s} \cap L_n),$$

and consequently in the critical points of F_n which lie in L_n (see Section 4). The Kirwan–Ness quotient admits a decomposition

$$L_n // GL(n) = X_{\alpha_1} \cup \cdots \cup X_{\alpha_s} \quad (\text{disjoint union}),$$

where each X_{α_i} is homeomorphic to $C_{\alpha_i}/U(n)$ and the following frontier property holds: there is a partial order on the indexing set $\{\alpha_1, \dots, \alpha_s\}$ such that

$$\bar{X}_\alpha \subset X_\alpha \cup \bigcup_{\beta > \alpha} X_\beta.$$

If $\alpha = (0; n)$ then X_α consists of finitely many points: the semisimple Lie algebras of dimension n .

We first study extremal points of $F_n : L_n \mapsto \mathbb{R}$, proving that the minimum value is attained at semisimple Lie algebras and the maximum value at the direct sum of the three-dimensional Heisenberg Lie algebra and the abelian algebra.

As expected, several strong compatibility properties between a Lie bracket μ and the fixed inner product $\langle \cdot, \cdot \rangle$ are necessary in order for $[\mu]$ to be a critical point of F_n . For instance, there must exist an orthonormal decomposition $\mathbb{C}^n = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$, with \mathfrak{h} a semisimple Lie subalgebra of μ , \mathfrak{a} abelian, $\mu(\mathfrak{h}, \mathfrak{a}) = 0$ and \mathfrak{n} the nilradical of μ , such that the adjoint action of the reductive part $\mathfrak{h} \oplus \mathfrak{a}$ of μ on the underlying inner product space $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is as nice as possible. More precisely,

- (i) $\text{ad}_\mu A$ is a normal operator (and so semisimple) for every $A \in \mathfrak{a}$.
- (ii) The real subalgebra $\mathfrak{k} = \{A \in \mathfrak{h} : (\text{ad}_\mu A)^* = -\text{ad}_\mu A\}$ is a maximal compact subalgebra of \mathfrak{h} , that is, $\mathfrak{h} = \mathfrak{k} + i\mathfrak{k}$.
- (iii) The hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h} \oplus \mathfrak{a}$ is given by

$$\langle A, B \rangle = -\frac{4}{c_\mu} \left(\frac{1}{2} \text{tr} \text{ad}_\mu A (\text{ad}_\mu B)^*|_{\mathfrak{h}} + \text{tr} \text{ad}_\mu A (\text{ad}_\mu B)^*|_{\mathfrak{n}} \right), \quad A, B \in \mathfrak{h} \oplus \mathfrak{a}.$$

We have obtained the following characterization of Lie algebras which are critical points of F_n (see Theorem 4.7 for a precise statement).

Theorem 1.1. *$[\mu] \in L_n$ is a critical point of F_n if and only if there exists an orthonormal decomposition $\mathbb{C}^n = \mathfrak{r} \oplus \mathfrak{n}$, where \mathfrak{r} is a reductive Lie subalgebra of μ with the properties that $(\text{ad}_\mu A)^* \in \text{Der}(\mu)$ for any $A \in \mathfrak{r}$, \mathfrak{n} is the nilradical of μ and $\mu|_{\mathfrak{n} \times \mathfrak{n}}$ is also a critical point of the corresponding F_m , $m = \dim \mathfrak{n}$.*

Thus, the classification of critical points of F_n in L_n reduces to the determination of those which are nilpotent. There is an intriguing interplay between nilpotent critical points and Riemannian geometry (see Remark 4.9).

In Section 5, the closed subset $\mathcal{A} \subset \mathcal{L}_n$ of Lie algebras having a codimension one abelian ideal is considered, in order to exemplify most of the notions studied in this paper. Finally, in Section 6, we consider for each type $\alpha = (k_1 < \dots < k_r; d_1, \dots, d_r)$ the action of the reductive Lie subgroup

$$\tilde{G}_\alpha = \left\{ g \in GL(d_1) \times \dots \times GL(d_r) : \prod_{i=1}^r (\det g_i)^{k_i} = \det g = 1 \right\}$$

on

$$V_\alpha = \{ \mu \in V_n : D_\alpha \in \text{Der}(\mu) \},$$

where D_α is the diagonal matrix with entries k_i and multiplicities d_i . If $\tilde{m} : \mathbb{P}V_\alpha \mapsto \mathfrak{t}_\alpha$ denotes the moment map for this action then $\tilde{m}^{-1}(0) = \mathbb{P}V_\alpha \cap C_\alpha$ and the semistable points $\mathbb{P}V_\alpha^{\text{ss}} = \mathbb{P}V_\alpha \cap S_\alpha$. Thus the categorical quotient $\mathbb{P}V_\alpha // \tilde{G}_\alpha$ coincides with $\mathbb{P}V_\alpha \cap C_\alpha / \tilde{K}_\alpha = C_\alpha / U(n)$, which parameterizes the critical points of type α , modulo isomorphism.

As an application, we study the $SL(n)$ -action on \mathcal{L}_n , obtaining that the semistable points are precisely the semisimple Lie algebras and moreover, we show that $SL(n) \cdot \mu$ is closed if and only if μ is semisimple. A new proof of the rigidity (i.e. $GL(n) \cdot \mu$ open in \mathcal{L}_n) of semisimple Lie algebras is also obtained. We finally give a complete description of the case $n = 4$.

2. A stratification of the null-cone

Let G be a complex reductive Lie group acting on a finite-dimensional complex vector space V , and let $X \subset V$ be a G -variety, that is, an algebraic variety which is G -invariant. The main problem of geometric invariant theory is to understand the orbit space of the action of G on X , parameterized by the quotient X/G (we refer to [18] for further information). The standard quotient topology of X/G can be very ugly, for instance, if x degenerates to y (i.e. $y \in \overline{G \cdot x}$) and $G \cdot x \neq G \cdot y$ then they cannot be separated by G -invariant open neighborhoods and so X/G is usually non-Hausdorff.

In order to avoid this problem one may consider a smaller quotient $X//G$ parameterizing only closed orbits. Mumford [16] proved that $X//G$ is again an algebraic variety; indeed, $X//G = \text{Spec}(\mathbb{C}[X]^G)$, that is, $X//G$ is the algebraic variety with coordinate ring $\mathbb{C}[X]^G$, the G -invariant polynomials on X . Consider $q : X \mapsto X//G$ the morphism of algebraic varieties determined by the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. $X//G$ is called the *categorical quotient* for the action of G on X since it satisfies the following universal property in the category of all algebraic varieties: for any morphism of algebraic varieties $\alpha : X \mapsto Y$ that is constant on G -orbits there exists a unique morphism $\beta : X//G \mapsto Y$ such that $\alpha = \beta \circ q$. The uniqueness of an object with such a property is clear from the definition. D. Luna proved that

actually, $X//G$ satisfies the same universal property in the category of all Hausdorff topological spaces. Recall that the usual quotient X/G would be the categorical quotient in the category of topological spaces.

The price to pay for a Hausdorff quotient is that in some cases $X//G$ classifies only a very few orbits. If we have for example that the homotheties $\{v \mapsto tv : t \in \mathbb{C}^*\}$ are contained in the action of G on V then $\{0\}$ will be the unique closed orbit, and hence $X//G$ will consist of only one point: too expensive. This is also clear from the fact that in such a case $\mathbb{C}[V]^G = \text{constant polynomials}$.

The aim of this section is to describe a wider quotient $X///G$, which of course is non-Hausdorff but it still satisfies some properties very similar in spirit to those of $X//G$. We call $X///G$ the *Kirwan–Ness quotient* for the action of G on X since its definition comes from some remarkable properties of the moment map for an action proved independently by Ness [17] and Kirwan [8], which we now overview.

Assume V is endowed with an hermitian inner product $\langle \cdot, \cdot \rangle$ which is invariant under the action of the maximal compact subgroup $K \subset G$. For each $v \in V$ define

$$\rho_v : G \mapsto \mathbb{R}, \quad \rho_v(g) = \|g.v\|^2 = \langle g.v, g.v \rangle.$$

Let $(d\rho_v)_e : \mathfrak{g} \mapsto \mathbb{R}$ denote the differential of ρ_v at the identity e of G . It follows from the K -invariance of $\langle \cdot, \cdot \rangle$ that $(d\rho_v)_e$ vanishes on \mathfrak{k} , and so we may view $(d\rho_v)_e \in \mathfrak{g}^*/\mathfrak{k}^*$, where \mathfrak{g}^* and \mathfrak{k}^* are the vector spaces of real-valued functionals on the Lie algebras of G and K , respectively. Since G is reductive $\mathfrak{g} = \mathfrak{k} + \mathfrak{i}\mathfrak{k}$, thus we may define a function

$$m : \mathbb{P}V \mapsto \mathfrak{i}\mathfrak{k}, \quad (m(x), A) = \frac{(d\rho_v)_e A}{\|v\|^2}, \quad 0 \neq v \in V, \quad x = [v], \quad (1)$$

where (\cdot, \cdot) is an $\text{Ad}(K)$ -invariant real inner product on $\mathfrak{i}\mathfrak{k}$ and $\mathbb{P}V$ is the projective space of lines in V . If $\pi : V \setminus \{0\} \mapsto \mathbb{P}V$ denotes the usual projection map, then $\pi(v) = [v] = x$. Under the natural identifications $\mathfrak{i}\mathfrak{k} = \mathfrak{i}\mathfrak{k}^* = \mathfrak{k}^*$, the function m is the moment map from symplectic geometry, corresponding to the Hamiltonian action of K on the symplectic manifold $\mathbb{P}V$ (see for instance the survey [9] or [16, Chapter 8]).

Definition 2.1. A vector $v \in V$ is said to be *unstable* if $0 \in \overline{G.v}$, the closure of the orbit $G.v$, and *semistable* if $0 \notin \overline{G.v}$. The set $N \subset V$ of unstable vectors is called the *null-cone* and the set of semistable vectors is denoted by V^{ss} .

The set V^{ss} is open in V . A well-known result due to Kempf and Ness [7] asserts that an orbit $G.v$ is closed if and only if $G.[v]$ meets $m^{-1}(0)$, and in that case the intersection coincides with $K.[v]$. For any $v \in V^{\text{ss}}$ there is a unique closed orbit in $\overline{G.v}$, and we then say that $v \sim w$ if the closures of their orbits contain the same closed orbit. If $\mathbb{P}V^{\text{ss}} = \pi(V^{\text{ss}})$, then the so-called *categorical quotient* $\mathbb{P}V//G = \mathbb{P}V^{\text{ss}}/\sim$ is homeomorphic to $m^{-1}(0)/K$, the symplectic quotient or reduced (phase) space of the symplectic manifold $\mathbb{P}V$ at the level 0.

Consider the functional square norm of the moment map

$$F : \mathbb{P}V \mapsto \mathbb{R}, \quad F(x) = \|m(x)\|^2 = (m(x), m(x)). \quad (2)$$

Thus, an orbit $G.v$ is closed if and only if $F(x) = 0$ for some $x \in G.[v]$, and in that case, the set of zeros of $F|_{G.x}$ coincides with $K.x$. Moreover, $\mathbb{P}V^{\text{ss}}$ is the set of points $[v] \in \mathbb{P}V$ with the property that the limit of the negative gradient flow of F is in $m^{-1}(0)$. A natural question arises: what is the role played by the remaining critical points of $F : \mathbb{P}V \mapsto \mathbb{R}$ (i.e. those for which $F(x) > 0$) in the study of the G -orbit space of the action of G on $\mathbb{P}V$, as well as on other complex projective G -varieties contained in $\mathbb{P}V$? This was precisely the aim of the paper [17] (see also [8]), where it is shown that the nonminimal critical points have influence in the study of the orbit space of the null-cone.

Theorem 2.2 (Ness [17]). *Let $F|_{G.x}$ denote the restriction of $F : \mathbb{P}V \mapsto \mathbb{R}$ to the G -orbit of $x \in \mathbb{P}V$. If x is a critical point of F then*

- (i) $F|_{G.x}$ attains its minimum value at x .
- (ii) $F|_{G.x}$ attains its minimum value only on the orbit $K.x$.

The nonminimal critical points (i.e. $F(x) > 0$) are all in the null-cone, that is, $0 \in \overline{G.v}$, $x = [v]$.

The proof of part (i) of the above theorem is based in the presently well-known convexity properties of moment maps (see [9]).

Theorem 2.3 (Ness [17]). *The negative gradient flow of $F : \mathbb{P}V \mapsto \mathbb{R}$ determines a stratification of the null-cone N . A stratum $S_{\langle h \rangle}$ of N , $h \in \mathfrak{if}$, is the set of all the points $x \in \mathbb{P}V$ which flow into $C_{\langle h \rangle}$, where $C_{\langle h \rangle}$ is the set of critical points y of F such that $m^*(y) \in \text{Ad}(K).h$. Moreover,*

- (i) *For each stratum $S_{\langle h \rangle}$ there exists a subspace $V_{\langle h \rangle} \subset V$ and a reductive subgroup $G_{\langle h \rangle} \subset G$ such that $C_{\langle h \rangle}/K = \mathbb{P}V_{\langle h \rangle}/G_{\langle h \rangle}$, and hence $C_{\langle h \rangle}/K$ is a projective algebraic variety.*
- (ii) *There are finitely many strata.*
- (iii) *Each stratum $S_{\langle h \rangle}$ is Zariski-locally closed, irreducible and nonsingular.*

Therefore, one may conclude that there exist also distinguished orbits in the null-cone N , namely those containing a critical point of F , which would play in some sense the same role as the closed orbits in V^{ss} . If C denotes the set of all critical points of the functional $F = \|m\|^2 : \mathbb{P}V \mapsto \mathbb{R}$, then it is natural to define the *Kirwan–Ness quotient* for the action of G on a projective G -variety $X \subset \mathbb{P}V$ by

$$X///G := C_X/K, \quad C_X = C \cap X.$$

It follows from Theorem 2.2(ii), that $X///G = GC_X/G$ and thus $X///G$ parameterizes precisely those G -orbits containing a critical point of F . We note that if the action of G on V contains the homotheties $\{v \mapsto tv : t \in \mathbb{C}^*\}$, then we may also define the Kirwan–Ness quotient for the action of G on an algebraic G -variety $X \subset V$ by

$$X///G = \{v \in X : [v] \in C\} / \mathbb{C}^*K,$$

and clearly $X///G = \pi(X)///G$. This new quotient $X///G$ is not a projective algebraic variety (not even Hausdorff) as in the case of $X//G$, but it still has some nice properties. Let \mathcal{B} denote the finite set indexing the strata described in Theorem 2.3. Consider the following partial order in \mathcal{B} : $\alpha < \beta$ if $F([v]) < F([w])$ for $[v] \in C_\alpha$, $[w] \in C_\beta$. We have that

$$\bar{S}_\alpha \subset S_\alpha \cup \bigcup_{\beta > \alpha} S_\beta,$$

and therefore the Kirwan–Ness quotient can be decomposed as the disjoint union

$$X///G = \bigcup_{\alpha \in \mathcal{B}} X_\alpha$$

of the projective algebraic varieties $X_\alpha = (C_\alpha \cap X)/K$, satisfying the following frontier property:

$$\bar{X}_\alpha \subset X_\alpha \cup \bigcup_{\beta > \alpha} X_\beta.$$

Recall that for $\alpha = \langle 0 \rangle$, we have $X_\alpha = \mathbb{P}V//G$, the categorical quotient.

Fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$. Thus, B determines a positive Weyl chamber $\mathfrak{it}^+ \subset \mathfrak{it}$, where \mathfrak{t} is the Lie algebra of T . Since $m : \mathbb{P}V \rightarrow \mathfrak{it}$ is K -equivariant, for any $x \in \mathbb{P}V$, $m(K.x)$ has a unique point of intersection $\check{m}(x)$ with \mathfrak{it}^+ , thus determining a function

$$\check{m} : \mathbb{P}V \rightarrow \mathfrak{it}^+.$$

Theorem 2.4 (Mumford [15]). *The image via \check{m} of any G -invariant closed set in $\mathbb{P}V$ is a rational convex polytope in \mathfrak{it}^+ , that is, the convex hull of a finite number of points with rational coordinates.*

We also have that $F(x)$ is precisely the square of the distance from the origin in \mathfrak{it} to the point $\check{m}(x)$ in the polytope $\check{m}(\overline{G.x})$.

3. The moment map for skew-symmetric algebras

The object of this section is to study the notions described in Section 2 for the $GL(n)$ -action on the vector space where Lie algebras live. Some of the results given here follow readily from the general case proved in [17], but since the adaptation can be sometimes rather difficult, we have argued directly in our particular situation for completeness.

Let $V_n = \Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ be the vector space of all alternating bilinear maps from $\mathbb{C}^n \times \mathbb{C}^n$ to \mathbb{C}^n , or in other words, the space of all skew-symmetric (nonassociative) algebras of dimension n . There is a natural action of $GL(n) = GL(n, \mathbb{C})$ on V_n given by

$$g.\mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y), \quad X, Y \in \mathbb{C}^n, \quad g \in GL(n), \quad \mu \in V_n. \quad (3)$$

We note that any $\mu \in V_n$ is in the null-cone, since for $g_t = tI$ we have that

$$\lim_{t \rightarrow 0} g_t^{-1}.\mu = \lim_{t \rightarrow 0} t\mu = 0,$$

and so $0 \in \overline{GL(n).\mu}$. The usual hermitian inner product $\langle \cdot, \cdot \rangle$ on \mathbb{C}^n , defines a $U(n)$ -invariant hermitian inner product on V_n , denoted also by $\langle \cdot, \cdot \rangle$, as follows:

$$\langle \mu, \lambda \rangle = \sum_{ijk} \langle \mu(X_i, X_j), X_k \rangle \overline{\langle \lambda(X_i, X_j), X_k \rangle}, \quad (4)$$

where $\{X_1, \dots, X_n\}$ is any orthonormal basis of \mathbb{C}^n . The Lie algebra of $GL(n)$ decomposes $\mathfrak{gl}(n) = \mathfrak{u}(n) + \mathfrak{iu}(n)$ in skew-hermitian and hermitian transformations, respectively, and an $\text{Ad}(U(n))$ -invariant hermitian inner product on $\mathfrak{gl}(n)$ is given by

$$(A, B) = \text{tr } AB^*, \quad A, B \in \mathfrak{gl}(n). \quad (5)$$

Thus we use (\cdot, \cdot) to identify $\mathfrak{iu}(n)$ with $\mathfrak{iu}(n)^*$. For each $\mu \in V_n$, consider the hermitian map $M_\mu \in \mathfrak{iu}(n)$ defined by

$$M_\mu = -4 \sum_i (\text{ad}_\mu X_i)^* \text{ad}_\mu X_i + 2 \sum_i \text{ad}_\mu X_i (\text{ad}_\mu X_i)^*, \quad (6)$$

where the adjoint map $\text{ad}_\mu X: \mathbb{C}^n \mapsto \mathbb{C}^n$ or left multiplication by X of the algebra μ is given, as usual, by $\text{ad}_\mu X(Y) = \mu(X, Y)$. It is a simple calculation to see that

$$\begin{aligned} \langle M_\mu X, Y \rangle &= -4 \sum_{ij} \langle \mu(X, X_i), X_j \rangle \overline{\langle \mu(Y, X_i), X_j \rangle} \\ &\quad + 2 \sum_{ij} \overline{\langle \mu(X_i, X_j), X \rangle} \langle \mu(X_i, X_j), Y \rangle \end{aligned} \quad (7)$$

for all $X, Y \in \mathbb{C}^n$. We will see below that the map $\mu \mapsto M_\mu$ is precisely the moment map for action (3). The action of $\mathfrak{gl}(n)$ on V_n obtained by differentiation of (3) is

given by

$$A \cdot \mu = -\delta_\mu(A) = A\mu(\cdot, \cdot) - \mu(A \cdot, \cdot) - \mu(\cdot, A \cdot), \quad A \in \mathfrak{gl}(n), \quad \mu \in V_n. \quad (8)$$

If $\mu \in V_n$ satisfies the Jacobi condition, then $\delta_\mu : \mathfrak{gl}(n) \mapsto V_n$ coincides with the cohomology coboundary operator of the Lie algebra (\mathbb{C}^n, μ) from level 1 to 2, relative to cohomology with values in the adjoint representation. Recall that $\text{Ker } \delta_\mu = \text{Der}(\mu)$, the Lie algebra of derivations of the algebra μ . The following technical lemma will be crucial in several proofs throughout the paper.

Lemma 3.1. *Let $p : \mathfrak{gl}(n) \mapsto \mathfrak{iu}(n)$ be the projection relative to the decomposition $\mathfrak{gl}(n) = \mathfrak{u}(n) + \mathfrak{iu}(n)$, and let $\delta_\mu^* : V_n \mapsto \mathfrak{gl}(n)$ denote the transpose of $\delta_\mu : \mathfrak{gl}(n) \mapsto V_n$, relative to the hermitian inner products (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$, on $\mathfrak{gl}(n)$ and V_n , respectively.*

(i) *If $M : V_n \mapsto \mathfrak{iu}(n)$ is defined by $M(\mu) = M_\mu$ for every $\mu \in V_n$, then*

$$(\text{dM})_\mu = -4p \circ \delta_\mu^*.$$

(ii) *$\text{tr } M_\mu D = 0$ for any $D \in \mathfrak{iu}(n) \cap \text{Der}(\mu)$.*

(iii) *$\text{tr } M_\mu[A, A^*] \geq 0$ for any $A \in \text{Der}(\mu)$. Equality holds if and only if we also have that $A^* \in \text{Der}(\mu)$.*

Proof. (i) Consider the line $\mu + t\lambda$ with $\mu, \lambda \in V_n$, $t \in \mathbb{R}$. Using (7), for any $A \in \mathfrak{iu}(n)$ we obtain that

$$\begin{aligned} ((\text{dM})_\mu(\lambda), A) &= \text{tr } A(\text{dM})_\mu(\lambda) = \sum_{pr} \langle (\text{dM})_\mu(\lambda) X_p, X_r \rangle \overline{\langle AX_p, X_r \rangle} \\ &= \sum_{pr} \frac{d}{dt} \Big|_{t=0} \langle M_{\mu+t\lambda} X_p, X_r \rangle \overline{\langle AX_p, X_r \rangle} \\ &= \sum_{pr} \left(\sum_{ij} -4 \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \mu(X_r, X_i), X_j \rangle} \right. \\ &\quad - 4 \langle \mu(X_p, X_i), X_j \rangle \overline{\langle \lambda(X_r, X_i), X_j \rangle} \\ &\quad + 2 \overline{\langle \lambda(X_i, X_j), X_p \rangle} \langle \mu(X_i, X_j), X_r \rangle \\ &\quad \left. + 2 \overline{\langle \mu(X_i, X_j), X_p \rangle} \langle \lambda(X_i, X_j), X_r \rangle \right) \overline{\langle AX_p, X_r \rangle}. \end{aligned}$$

We now interchange, in even lines, the indexes p and r , obtaining that

$$\begin{aligned} ((\text{dM})_\mu(\lambda), A) &= \text{Re} \sum_{prij} -8 \overline{\langle \mu(X_r, X_i), X_j \rangle} \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle AX_p, X_r \rangle} \\ &\quad + 4 \overline{\langle \mu(X_i, X_j), X_r \rangle} \langle \lambda(X_i, X_j), X_p \rangle \overline{\langle AX_p, X_r \rangle}. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
 \langle \lambda, \delta_\mu(A) \rangle &= \sum_{pij} \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \delta_\mu(A)(X_p, X_i), X_j \rangle} \\
 &= \sum_{pij} \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \mu(AX_p, X_i), X_j \rangle} \\
 &\quad + \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \mu(X_p, AX_i), X_j \rangle} \\
 &\quad - \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle A\mu(X_p, X_i), X_j \rangle} \\
 &= \sum_{pijr} \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \mu(X_r, X_i), X_j \rangle} \overline{\langle AX_p, X_r \rangle} \\
 &\quad + \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \mu(X_p, X_r), X_j \rangle} \overline{\langle AX_i, X_r \rangle} \\
 &\quad - \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \mu(X_p, X_i), X_r \rangle} \overline{\langle AX_j, X_r \rangle}.
 \end{aligned}$$

By interchanging the indexes p and i in the second line, and p and j in the third one, we get

$$\begin{aligned}
 \langle \lambda, \delta_\mu(A) \rangle &= 2 \sum_{pij} \langle \lambda(X_p, X_i), X_j \rangle \overline{\langle \mu(X_r, X_i), X_j \rangle} \overline{\langle AX_p, X_r \rangle} \\
 &\quad - \langle \lambda(X_i, X_j), X_p \rangle \overline{\langle \mu(X_i, X_j), X_r \rangle} \overline{\langle AX_p, X_r \rangle}.
 \end{aligned}$$

We then can deduce from the two computations above that

$$((dM)_\mu(\lambda), A) = -4 \operatorname{Re} \langle \lambda, \delta_\mu(A) \rangle = -4 \operatorname{Re}(\delta_\mu^*(\lambda), A) = \operatorname{Re}(-4p \circ \delta_\mu^*(\lambda), A),$$

for every $A \in \mathfrak{iu}(n)$, which concludes the proof of (i).

(ii) Using that $2M_\mu = \frac{d}{dt}|_0 (1+t)^2 M_\mu = \frac{d}{dt}|_0 M_{\mu+t\mu} = (dM)_\mu(\mu)$ and part (i), we obtain that

$$\begin{aligned}
 \operatorname{tr} M_\mu D &= \frac{1}{2} \operatorname{tr}(dM)_\mu(\mu) D = -2 \operatorname{tr} p \circ \delta_\mu^*(\mu) D \\
 &= -2 \operatorname{Re}(\delta_\mu^*(\mu), D) = -2 \operatorname{Re} \langle \mu, \delta_\mu(D) \rangle = 0.
 \end{aligned}$$

(iii) It follows easily from the K -invariance of $\langle \cdot, \cdot \rangle$ on V_n that

$$\langle A \cdot \mu, \lambda \rangle = \langle \mu, A^* \cdot \lambda \rangle$$

for any $A \in \mathfrak{gl}(n)$. We then have by part (i) that

$$\begin{aligned}
 \operatorname{tr} M_\mu[A, A^*] &= \frac{1}{2} \operatorname{tr}(dM)_\mu(\mu)[A, A^*] = -2 \operatorname{Re}(\delta_\mu^*(\mu), [A, A^*]) \\
 &= -2 \operatorname{Re} \langle \mu, \delta_\mu([A, A^*]) \rangle = 2 \operatorname{Re} \langle \mu, [A, A^*] \cdot \mu \rangle
 \end{aligned}$$

$$\begin{aligned}
&= 2 \operatorname{Re} \langle \mu, A.(A^*.\mu) - A^*.(A.\mu) \rangle = 2 \operatorname{Re} \langle \mu, A.(A^*.\mu) \rangle \\
&= 2 \langle A^*.\mu, A^*.\mu \rangle \geq 0,
\end{aligned}$$

and it equals 0 if and only if $A^*.\mu = 0$, that is, $A^* \in \operatorname{Der}(\mu)$. \square

We now calculate the moment map and the gradient of the functional square norm of the moment map, obtaining a rather computable characterization of their critical points, which will be very useful.

Proposition 3.2. *The moment map $m: \mathbb{P}V_n \mapsto \mathfrak{iu}(n)$, the functional square norm of the moment map $F_n = \|m\|^2: \mathbb{P}V_n \mapsto \mathbb{R}$ and the gradient of F_n are, respectively, given by*

$$m([\mu]) = \mathbf{M}_\mu, \quad F_n([\mu]) = \operatorname{tr} \mathbf{M}_\mu^2, \quad \operatorname{grad}(F_n)_{[\mu]} = -8\pi_* \delta_\mu(\mathbf{M}_\mu), \quad \|\mu\| = 1, \quad (9)$$

where π_* denotes the derivative of $\pi: V_n \setminus \{0\} \mapsto \mathbb{P}V_n$, the canonical projection. Moreover, the following statements are equivalent:

- (i) $[\mu] \in \mathbb{P}V_n$ is a critical point of F_n .
- (ii) $[\mu] \in \mathbb{P}V_n$ is a critical point of $F_n|_{\operatorname{GL}(n).[\mu]}$.
- (iii) $\mathbf{M}_\mu = cI + D$ for some $c \in \mathbb{R}$ and $D \in \operatorname{Der}(\mu)$.

Proof. For any $A \in \mathfrak{iu}(n)$ we have that

$$(\mathrm{d}\rho_\mu)_I(A) = \frac{\mathrm{d}}{\mathrm{d}t} \big|_0 \langle e^{tA}.\mu, e^{tA}.\mu \rangle = -2 \operatorname{Re} \langle \delta_\mu(A), \mu \rangle = -2 \operatorname{Re}(A, \delta_\mu^*(\mu)).$$

Now, using Lemma 3.1(i), we get

$$(\mathrm{d}\rho_\mu)_I(A) = -2 \operatorname{Re} \langle A, -\frac{1}{4}(\mathrm{d}\mathbf{M})_\mu(\mu) \rangle = \operatorname{Re}(A, \mathbf{M}_\mu) = \operatorname{Re} \operatorname{tr} \mathbf{M}_\mu A,$$

which proves the formula for m . The first assertion on F_n is self-evident. To prove the second one, we only need to compute the gradient of $F_n: V_n \mapsto \mathbb{R}$, $F_n(\mu) = \operatorname{tr} \mathbf{M}_\mu^2$, and then to project it via π_* . If $\mu, \lambda \in V_n$, then it follows from Lemma 3.1(i), that

$$\begin{aligned}
\operatorname{Re} \langle \operatorname{grad}(F_n)_\mu, \lambda \rangle &= \frac{\mathrm{d}}{\mathrm{d}t} \big|_0 F_n(\mu + t\lambda) = \frac{\mathrm{d}}{\mathrm{d}t} \big|_0 \operatorname{tr} \mathbf{M}_{\mu+t\lambda}^2 = 2 \operatorname{Re} \operatorname{tr} \left(\frac{\mathrm{d}}{\mathrm{d}t} \big|_0 \mathbf{M}_{\mu+t\lambda} \right) \mathbf{M}_\mu \\
&= 2 \operatorname{Re}((\mathrm{d}\mathbf{M}_\mu)_\mu(\lambda), \mathbf{M}_\mu) = -8 \operatorname{Re}(\delta_\mu^*(\lambda), \mathbf{M}_\mu) \\
&= -8 \operatorname{Re} \langle \lambda, \delta_\mu(\mathbf{M}_\mu) \rangle.
\end{aligned}$$

This implies that $\operatorname{grad}(F_n)_\mu = -8\delta_\mu(\mathbf{M}_\mu)$, concluding the proof of (9).

The equivalence between (i) and (ii) has been observed in [17], and it follows from the fact that $\operatorname{grad}(F_n)_{[\mu]} \in \operatorname{T}_{[\mu]} \operatorname{GL}(n).[\mu]$ for any $[\mu] \in \mathbb{P}V_n$. Indeed, by

(9) we have that

$$\text{grad}(F_n)_{[\mu]} = 8\pi_* \left(\frac{d}{dt} \Big|_0 e^{tM_\mu} \cdot \mu \right) = 8 \frac{d}{dt} \Big|_0 e^{tM_\mu} \cdot [\mu].$$

In order to prove that (i) is equivalent to (iii), recall first that $\text{Ker } \pi_*(\mu) = \mathbb{C}\mu$. Hence we obtain from (9) that $[\mu]$ is a critical point of F_n if and only if $\delta_\mu(M_\mu) \in \mathbb{C}\mu$, or equivalently, $M_\mu \in \mathbb{C}I \oplus \text{Der}(\mu)$, since $\delta_\mu(I) = \mu$ and $\text{Ker } \delta_\mu = \text{Der}(\mu)$ (see (8)). If $M_\mu = cI + D$ with $c \in \mathbb{C}$ and $D \in \text{Der}(\mu)$, then it is evident that D is normal, and so D^* is also a derivation of μ (see Lemma 3.1(iii)). Let $D = D_s + D_h$ be the skew-hermitian and hermitian parts of D . Since $D_s = -i \text{Im}(c)I$ has to be a derivation we get that $D_s = 0$, and therefore $M_\mu = cI + D_h$ and $c \in \mathbb{R}$, as it was to be shown. \square

In the frame of skew-symmetric algebras, the result due to Ness given in Theorem 2.2 can be stated as follows.

Theorem 3.3 (Ness [17]). *If $[\mu]$ is a critical point of the functional $F_n: \mathbb{P}V_n \mapsto \mathbb{R}$ given by $F_n([\lambda]) = \text{tr } M_\lambda^2$ ($\|\lambda\| = 1$), then*

- (i) $F_n|_{GL(n) \cdot [\mu]}$ attains its minimum value at $[\mu]$.
- (ii) $[\lambda] \in GL(n) \cdot [\mu]$ is a critical point of F_n if and only if $[\lambda] \in U(n) \cdot [\mu]$.

We now describe some particular features of the critical points and the stratification for our case.

Lemma 3.4. *Let $[\mu] \in \mathbb{P}V_n$ be a critical point of F_n , say $M_\mu = c_\mu I + D_\mu$ for some $c_\mu \in \mathbb{R}$ and $D_\mu \in \text{Der}(\mu)$. Then $c_\mu = \frac{\text{tr } M_\mu^2}{\text{tr } M_\mu} = -\frac{1}{2} \frac{\text{tr } M_\mu^2}{\|\mu\|^2}$, and if $D_\mu \neq 0$ then $\text{tr } D_\mu > 0$ and $c_\mu = -\frac{\text{tr } D_\mu^2}{\text{tr } D_\mu}$.*

Proof. Both assertions follow from the fact that $\text{tr } M_\mu D_\mu = 0$ (see Lemma 3.1(ii)) and $\text{tr } M_\mu = -2\|\mu\|^2$. \square

The proof of the following rationality result, which is just a bit stronger than the one given in [17, Section 4] for the general case, is based on the proof by Heber of [6, Theorem 4.14].

Theorem 3.5. *Let $[\mu] \in \mathbb{P}V_n$ be a critical point of F_n , with $M_\mu = c_\mu I + D_\mu$ for some $c_\mu \in \mathbb{R}$ and $D_\mu \in \text{Der}(\mu)$. Then there exists $c > 0$ such that the eigenvalues of cD_μ are all nonnegative integers prime to each other, say $k_1 < \dots < k_r \in \mathbb{Z}_{\geq 0}$ with multiplicities $d_1, \dots, d_r \in \mathbb{N}$.*

Proof. If $D_\mu = 0$ then there is nothing to prove, so we assume $D_\mu \neq 0$. Consider $\mathbb{C}^n = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ the orthogonal decomposition in eigenspaces of D_μ , say

$D_\mu|_{\mathfrak{g}_i} = c_i I_{\mathfrak{g}_i}$ with $c_1 < \dots < c_r$. Since $D_\mu \in \text{Der}(\mu)$ we have that $\mu(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}_k$ if and only if $c_i + c_j = c_k$; otherwise $\mu(\mathfrak{g}_i, \mathfrak{g}_j) = 0$. A crucial point here is that any hermitian map A of \mathbb{C}^n defined by $A|_{\mathfrak{g}_i} = a_i I_{\mathfrak{g}_i}$, satisfying $a_i + a_j = a_k$ for all i, j, k such that $c_i + c_j = c_k$, is also a derivation of μ . Now, using Lemmas 3.1(ii) and 3.4, we get

$$\text{tr } D_\mu A = c_\mu \text{tr } A = - \frac{\text{tr } D_\mu^2}{\text{tr } D_\mu} \text{tr } A \quad (10)$$

for every A satisfying the above conditions. In other words, if $\{e_1, \dots, e_r\}$ denotes the canonical basis of \mathbb{R}^r and $\alpha = \sum c_i^2 / \sum c_i$, then (10) says that the vector $(c_1 - \alpha, \dots, c_r - \alpha)$ is orthogonal to F^\perp , where

$$F = \{e_i + e_j - e_k : c_i + c_j = c_k\}.$$

This implies that $(c_1 - \alpha, \dots, c_r - \alpha) \in \langle F \rangle$ (subspace of \mathbb{R}^r linearly spanned by F), and thus

$$(c_1 - \alpha, \dots, c_r - \alpha) = \sum_{p=1}^s b_p (e_{i_p} + e_{j_p} - e_{k_p}), \quad b_p \in \mathbb{R},$$

where $\{e_{i_p} + e_{j_p} - e_{k_p} : p = 1, \dots, s\}$ is a basis of $\langle F \rangle$. We now consider the $(s \times r)$ -matrix

$$E = \begin{bmatrix} e_{i_1} + e_{j_1} - e_{k_1} \\ \vdots \\ e_{i_s} + e_{j_s} - e_{k_s} \end{bmatrix}, \quad \text{so that } EE^t \in GL(s, \mathbb{Q}),$$

and furthermore

$$E \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} = 0, \quad E \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad E^t \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix} = (c_1 - \alpha, \dots, c_r - \alpha).$$

This implies that

$$\frac{1}{\alpha} (c_1, \dots, c_r) = (1, \dots, 1) - E^t (EE^t)^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Q}^r \quad (11)$$

as it was to be shown. It remains to prove that $0 \leq c_1$. If $D_\mu X = c_1 X$ then $c_1 \text{ad}_\mu X = \text{ad}_\mu D_\mu X = [D_\mu, \text{ad}_\mu X]$, and therefore

$$c_1 \text{tr ad}_\mu X (\text{ad}_\mu X)^* = \text{tr } D_\mu [\text{ad}_\mu X, (\text{ad}_\mu X)^*] = \text{tr } M_\mu [\text{ad}_\mu X, (\text{ad}_\mu X)^*] \geq 0,$$

by Lemma 3.1(iii). If we assume $\text{ad}_\mu X \neq 0$, we obtain that $c_1 \geq 0$. Otherwise, if $\text{ad}_\mu X = 0$ then $0 \leq \langle M_\mu X, X \rangle = (c_\mu + c_1) \langle X, X \rangle$, and so $c_1 > 0$ since $c_\mu < 0$ by Lemma 3.4. \square

Definition 3.6. The data set $(k_1 < \dots < k_r; d_1, \dots, d_r)$ in the above theorem is called the *type* of the critical point $[\mu]$.

Proposition 3.7. *In any fixed dimension n , there are only finitely many types of critical points of $F_n: \mathbb{P}V_n \mapsto \mathbb{R}$.*

Proof. Eq. (11) says that the numbers k_1, \dots, k_r can be recovered from the knowledge of r and the finite set F_n alone. Thus, the finiteness of the types follows from the finiteness of the partitions $n = d_1 + \dots + d_r$ and the different sets F_n . \square

Since the null-cone of V_n is all of V_n , the strata $S_{\langle h \rangle}$'s described in Theorem 2.3 determines a stratification of V_n . The set of types of critical points is in bijection with the set of strata. Indeed, for a type $\alpha = (k_1 < \dots < k_r; d_1, \dots, d_r)$ we define

$$h_\alpha = -\frac{k_1^2 d_1 + \dots + k_r^2 d_r}{k_1 d_1 + \dots + k_r d_r} I + \begin{bmatrix} k_1 I_{d_1} & & \\ & \ddots & \\ & & k_r I_{d_r} \end{bmatrix} \in \mathfrak{iu}(n),$$

where I_{d_i} denotes the $d_i \times d_i$ identity matrix. Thus the set $C_\alpha = C_{\langle h_\alpha \rangle}$ given in Theorem 2.3 is precisely the set of critical points $[\mu]$ of $F_n: \mathbb{P}V_n \mapsto \mathbb{R}$ such that $m([\mu]) = M_\mu = c_\mu I + D_\mu$ is conjugate to h_α , or equivalently,

$$C_\alpha = \{[\mu] \in \mathbb{P}V_n : [\mu] \text{ is a critical point of } F_n \text{ of type } \alpha\}.$$

This implies that each stratum is of the form $S_\alpha = S_{\langle h_\alpha \rangle}$ for some type α , where

$$S_\alpha = \left\{ [\lambda] \in \mathbb{P}V_n : \begin{array}{l} \text{the limit of the } -\text{grad}(F_n) \\ \text{flow starting from } [\lambda] \text{ is in } C_\alpha \end{array} \right\}. \quad (12)$$

If $\alpha_1, \dots, \alpha_r$ are the different types of critical points then

$$\mathbb{P}V_n = S_{\alpha_1} \cup \dots \cup S_{\alpha_s}$$

determines a stratification of $\mathbb{P}V_n$, for which each stratum S_{α_i} is locally closed, irreducible and nonsingular. If $\check{m}: \mathbb{P}V_n \mapsto \mathfrak{iu}(n)^+$ is the function considered in Theorem 2.4, then $\check{m}([\mu]) = h_\alpha$ for any critical point $[\mu]$ of type α .

Since $\text{grad}(F_n)$ is always tangent to $GL(n)$ -orbits, we have that the stratification of any $GL(n)$ -invariant projective algebraic variety $X \subset \mathbb{P}V_n$ is obtained just by

intersecting the stratum S_α of $\mathbb{P}V_n$ with X . In view of the equivalence between (i) and (ii) in Proposition 3.2, the critical points of $F_n: X \mapsto \mathbb{R}$ are precisely the critical points of $F_n: \mathbb{P}V_n \mapsto \mathbb{R}$ which lie in X . Thus, we will often refer to them just as critical points of F_n .

The following two propositions on the critical values of F_n and abelian factors were proved in [14, Section 3].

Proposition 3.8. *Let $[\mu] \in \mathbb{P}V_n$ be a critical point of F_n of type $(k_1 < \dots < k_r; d_1, \dots, d_r)$, different from $(0, n)$. Then,*

$$F_n([\mu]) = 4 \left(n - \frac{(k_1 d_1 + \dots + k_r d_r)^2}{k_1^2 d_1 + \dots + k_r^2 d_r} \right)^{-1}.$$

Proposition 3.9. *Let $[\mu] \in \mathbb{P}V_n$ and $[\lambda] \in \mathbb{P}V_m$ be critical points of F_n and F_m , respectively. Then the direct sum $[\mu \oplus c\lambda] \in \mathbb{P}V_{n+m}$ is a critical point of F_{n+m} for a suitable $c \in \mathbb{R}$. If in addition λ is abelian, then $F_n([\mu]) = F_{n+m}([\mu \oplus \lambda])$ and the type of $[\mu \oplus \lambda]$ is given by*

$$\left(ak_1 < \dots < \frac{k_1^2 d_1 + \dots + k_r^2 d_r}{d} < \dots < ak_r; d_1, \dots, m, \dots, d_r \right),$$

where $d = \text{mcd}(k_1 d_1 + \dots + k_r d_r, k_1^2 d_1 + \dots + k_r^2 d_r)$ and $a = \frac{k_1 d_1 + \dots + k_r d_r}{d}$. In case that $\frac{k_1^2 d_1 + \dots + k_r^2 d_r}{d} = ak_i$ for some i , then the multiplicity is $m + d_i$.

4. Critical points in the variety of Lie algebras

The space of all n -dimensional complex Lie algebras can be naturally identified with the subset $\mathcal{L}_n \subset V_n$ of all Lie brackets. \mathcal{L}_n is an algebraic set, since the Jacobi identity is given by polynomial conditions. The isomorphism class of a Lie algebra $\mu \in \mathcal{L}_n$ is then the orbit $\text{GL}(n) \cdot \mu$ under the ‘change of basis’ action of $\text{GL}(n)$ on \mathcal{L}_n given in (3).

Definition 4.1. We say that μ *degenerates to* λ if $\lambda \in \overline{\text{GL}(n) \cdot \mu}$, which will be often denoted by $\mu \mapsto \lambda$.

Every degeneration will be assumed to be nontrivial, that is, λ lies in the boundary of $\text{GL}(n) \cdot \mu$. If $\mu \mapsto \lambda$ then we may say roughly that λ is ‘more abelian’ than μ ; in fact, $\dim \text{Der}(\lambda) > \dim \text{Der}(\mu)$, $\dim \lambda(\mathbb{C}^n, \mathbb{C}^n) \leq \dim \mu(\mathbb{C}^n, \mathbb{C}^n)$, $\dim \mathfrak{z}(\lambda) \geq \dim \mathfrak{z}(\mu)$ and $\text{ab}(\lambda) \geq \text{ab}(\mu)$, where $\mathfrak{z}(\mu)$ denotes the center of μ and $\text{ab}(\mu)$ is the dimension of a maximal abelian subalgebra of μ (see [2]).

We note that the $-\text{grad}(F_n)$ flow $\mu(t)$ defines a degeneration (possibly trivial) of the starting point $\mu(0)$ to a critical point of F_n , since $\mu(t) \in \text{GL}(n) \cdot \mu(0)$ for any t . Therefore, such a distinguished degeneration associates to each Lie algebra $\mu = \mu(0)$ a $\mathbb{Z}_{\geq 0}$ -graded Lie algebra $\lambda = \lim_{t \rightarrow 0} \mu(t)$, and there are finitely many possible

gradations. It is easy to see that if μ is nilpotent then the associated λ is actually \mathbb{N} -graded.

Let L_n be the projective algebraic variety obtained by projectivization of the algebraic variety $\mathcal{L}_n \subset V_n$ of n -dimensional Lie algebras, that is, $L_n = \pi(\mathcal{L}_n \setminus \{0\})$. We are interested in the Kirwan–Ness quotient $L_n // GL(n)$, and thus we want to understand the critical points of F_n which lie in L_n , where F_n is the functional square norm of the moment map (see Section 3). A first natural question would be to find the global minima and maxima of $F_n : L_n \mapsto \mathbb{R}$.

Lemma 4.2. *$[\lambda]$ is a critical point of F_n of type $(0; n)$ (i.e. $M_\lambda \in \mathbb{R}I$) if and only if $F_n([\lambda]) = \frac{4}{n}$. In that case, $F_n : \mathbb{P}V_n \mapsto \mathbb{R}$ attains its minimum at $[\lambda]$.*

Proof. It follows from the formula

$$F_n([\mu]) = 4 \frac{\text{tr } M_\mu^2}{(\text{tr } M_\mu)^2}, \quad [\mu] \in \mathbb{P}V_n,$$

and a standard analysis of the function $f : \mathbb{R}^n \mapsto \mathbb{R}$, $f(c_1, \dots, c_n) = c_1^2 + \dots + c_n^2 / (c_1 + \dots + c_n)^2$.

Theorem 4.3. *Assume there exists a semisimple Lie algebra of dimension n . Then $F_n : L_n \mapsto \mathbb{R}$ attains its minimum value at a point $[\lambda] \in GL(n) \cdot [\mu]$ if and only if μ is semisimple. In such a case, $F_n([\lambda]) = \frac{4}{n}$.*

Proof. If μ is semisimple, then by standard methods we can get a basis $\{Y_i\}$ of \mathbb{C}^n which is orthonormal with respect to a suitable hermitian inner product $\langle \cdot, \cdot \rangle'$, such that the ‘Casimir’ map satisfies $\sum (\text{ad}_\mu Y_i)^2 \in \mathbb{R}I$ and $\text{ad}_\mu Y_i$ is skew-hermitian relative to $\langle \cdot, \cdot \rangle'$ for all i . Define $g \in GL(n)$ by $gY_i = X_i$ and consider $\lambda = g \cdot \mu$, where $\{X_i\}$ is any orthonormal basis of \mathbb{C}^n relative to our fixed inner product $\langle \cdot, \cdot \rangle$. We note that for any transformation T of \mathbb{C}^n , the adjoints of T relative to $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are related by $(gTg^{-1})^* = gT^{*'}g^{-1}$. Thus the moment map at λ is given by

$$\begin{aligned} M_\lambda &= -4 \sum_i (\text{ad}_\lambda X_i)^* \text{ad}_\lambda X_i + 2 \sum_i \text{ad}_\lambda X_i (\text{ad}_\lambda X_i)^* \\ &= -4 \sum_i (g \text{ad}_\mu (g^{-1} X_i) g^{-1})^* g \text{ad}_\mu (g^{-1} X_i) g^{-1} \\ &\quad + 2 \sum_i g \text{ad}_\mu (g^{-1} X_i) g^{-1} (g \text{ad}_\mu (g^{-1} X_i) g^{-1})^* \\ &= -4 \sum_i g (\text{ad}_\mu Y_i)^{*'} g^{-1} g \text{ad}_\mu Y_i g^{-1} + 2 \sum_i g \text{ad}_\mu Y_i g^{-1} g (\text{ad}_\mu Y_i)^{*'} g^{-1} \\ &= 2g \sum_i (\text{ad}_\mu Y_i)^2 g^{-1} \in \mathbb{R}I. \end{aligned}$$

This means that $[\lambda] = [g.\mu] \in GL(n).[\mu]$ is a critical point of F_n of type $(0; n)$, and thus $[\lambda]$ is a minimum of F_n (see Lemma 4.2).

Conversely, assume $[\lambda] \in GL(n).[\mu]$ is a minimum of $F_n : L_n \mapsto \mathbb{R}$. Thus $M_\lambda = cI$ for some $c < 0$ (see Lemma 4.2). It is clear that μ is semisimple if and only if λ is so. Let \mathfrak{s}_λ denote the radical of λ and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}_\lambda$ the orthogonal decomposition. We also decompose orthogonally $\mathfrak{s}_\lambda = \mathfrak{a} \oplus \mathfrak{n}_\lambda = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}_\lambda$, where $\mathfrak{n}_\lambda = \lambda(\mathfrak{s}_\lambda, \mathfrak{s}_\lambda)$ and \mathfrak{z}_λ is the center of the nilpotent Lie algebra $(\mathfrak{n}_\lambda, \lambda|_{\mathfrak{n}_\lambda \times \mathfrak{n}_\lambda})$. Suppose that $\mathfrak{z}_\lambda \neq 0$ and consider an orthonormal basis $\{H_i\}$, $\{A_i\}$, $\{V_i\}$ and $\{Z_i\}$ of \mathfrak{h} , \mathfrak{a} , \mathfrak{v} , \mathfrak{z}_λ , respectively. If $\{X_i\} = \{H_i\} \cup \{A_i\} \cup \{V_i\} \cup \{Z_i\}$ then for every $Z \in \mathfrak{z}_\lambda$,

$$\begin{aligned} 0 > \langle M_\lambda Z, Z \rangle &= -4 \sum_{ij} |\langle \lambda(Z, X_i), X_j \rangle|^2 + 2 \sum_{ij} |\langle \lambda(X_i, X_j), Z \rangle|^2 \\ &= -4 \sum_{ij} |\langle \lambda(Z, H_i), Z_j \rangle|^2 - 4 \sum_{ij} |\langle \lambda(Z, A_i), Z_j \rangle|^2 \\ &\quad + 4 \sum_{ij} |\langle \lambda(Z, V_i), Z_j \rangle|^2 + 4 \sum_{ij} |\langle \lambda(Z, Z_i), Z_j \rangle|^2 + \beta(Z), \end{aligned}$$

where $\beta(Z) = 2 \sum_{ij} |\langle \lambda(X_i, X_j), Z \rangle|^2 \geq 0$, $X_i, X_j \in \{H_i\} \cup \{A_i\} \cup \{V_i\}$ (note that both $\text{ad}_\lambda \mathfrak{h}$ and $\text{ad}_\lambda \mathfrak{a}$ leave invariant \mathfrak{z}_λ). This implies that

$$0 > \sum_k \langle M_\lambda Z_k, Z_k \rangle = \sum_k \beta(Z_k) \geq 0,$$

which is a contradiction. Thus \mathfrak{z}_λ has to be $\{0\}$ and hence \mathfrak{s}_λ is abelian. Now, by applying the same argument to a nonzero $A \in \mathfrak{s}_\lambda$ we also get a contradiction, which implies that $\mathfrak{s}_\lambda = 0$. Therefore λ is semisimple. \square

Remark 4.4. The second part of the above proof implies that any $\mu \in \mathcal{L}_n$ for which there exists $\lambda \in GL(n).\mu$ such that all eigenvalues of M_λ are negative, must be semisimple. In particular, $GL(n).\mu$ contains a critical point of type $(0; n)$ if and only if μ is semisimple. Moreover, the stratum $S_{(0; n)} \cap L_n$ is precisely the set of semisimple Lie brackets. Indeed, for any $[\lambda] \in S_{(0; n)}$, the $-\text{grad}(F_n)$ flow $\{\lambda(t)\} \subset GL(n).[\lambda]$ starting from $[\lambda]$ converges to a critical point of type $(0; n)$ and so the eigenvalues of $M_{\lambda(t)}$ will be negative for sufficiently large t .

Proposition 4.5. Assume there does not exist a semisimple Lie algebra of dimension n .

- (i) If there exists an $(n-1)$ -dimensional semisimple Lie algebra, then $F_n : L_n \mapsto \mathbb{R}$ attains its minimum value at some point in $GL(n).[\mu]$ if and only if μ is isomorphic to a reductive Lie algebra with one-dimensional center.
- (ii) If there is no any $(n-1)$ -dimensional semisimple Lie algebra, then $F_n : L_n \mapsto \mathbb{R}$ attains its minimum value at some point in $GL(n).[\mu]$ if and only if μ is isomorphic to the direct sum of an $(n-2)$ -dimensional semisimple Lie algebra and the only two-dimensional non-abelian Lie algebra.

In both cases, the type is $(0 < 1; n-1, 1)$ and the minimum value equals $\frac{4}{n-1}$.

Proof. It follows from Proposition 3.9 that the orbits considered in both cases contain a critical point of F_n of type $(0 < 1; n-1, 1)$. Thus the critical value is $\frac{4}{n-1}$ by Proposition 3.8. It then suffices to prove that $F_n([\mu]) > \frac{4}{n-1}$ for any critical point $[\mu] \in L_n$ of type $(k_1 < \dots < k_r; d_1, \dots, d_r)$ different from $(0; n)$, as there is no any semisimple Lie algebra of dimension n (see Remark 4.4). Actually, we will not use the fact that $\mu \in \mathcal{L}_n$.

We have that $k_1 d_1 + \dots + k_r d_r > k_r$, therefore

$$\frac{k_1 d_1}{\sum k_i d_i} k_1 + \dots + \frac{k_r d_r}{\sum k_i d_i} k_r \leq k_r < k_1 d_1 + \dots + k_r d_r.$$

This implies that $k_1^2 d_1 + \dots + k_r^2 d_r < (k_1 d_1 + \dots + k_r d_r)^2$, and hence we obtain from Proposition 3.8 that

$$F_n([\mu]) = 4 \left(n - \frac{(k_1 d_1 + \dots + k_r d_r)^2}{k_1^2 d_1 + \dots + k_r^2 d_r} \right)^{-1} < \frac{4}{n-1},$$

concluding the proof. \square

If μ is semisimple then the minimum $[\lambda] \in \text{GL}(n) \cdot [\mu]$ of $F_n : L_n \mapsto \mathbb{R}$ is also a minimum of $F_n : \mathbb{P}V_n \mapsto \mathbb{R}$ (see Lemma 4.2). On the contrary, the minima given in Proposition 4.5 are not minima of $F_n : \mathbb{P}V_n \mapsto \mathbb{R}$. In fact, if $[\lambda] \in \mathbb{P}\mathcal{L}_{n-z}$ ($z = 1$ or 2) is a semisimple critical point such that $M_\lambda = -12(n-z)I$, then it is easy to check that the bilinear form $\mu \in V_n$, where we consider $\mathbb{C}^n = \mathbb{C}^z \oplus \mathbb{C}^{n-z}$, defined by

$$\mu(X_1, X) = X \quad \forall X \in \mathbb{C}^{n-z}, \quad \mu|_{\mathbb{C}^{n-z} \times \mathbb{C}^{n-z}} = \lambda,$$

$$\mu(X_1, X_2) = (n-2)^{\frac{1}{2}}(X_1 + X_2) \quad (\text{if } z = 2),$$

is a critical point of $F_n : \mathbb{P}V_n \mapsto \mathbb{R}$ of type $(0; n)$. Consequently $F_n([\mu]) = \frac{4}{n}$, and so $[\mu]$ is a minimum of $F_n : \mathbb{P}V_n \mapsto \mathbb{R}$ by Lemma 4.2.

We now study the maxima of $F_n : L_n \mapsto \mathbb{R}$. Let $\mu_{\text{he}}, \mu_{\text{hy}}$ denote the Lie brackets defined by

$$\mu_{\text{he}}(X_1, X_2) = X_3, \quad \mu_{\text{hy}}(X_1, X_i) = X_i, \quad i = 2, \dots, n,$$

and zero otherwise. Note that μ_{he} is isomorphic to the direct sum of the three-dimensional Heisenberg Lie algebra and an abelian Lie algebra.

Theorem 4.6. *The functional $F_n : L_n \mapsto \mathbb{R}$ attains its maximum value at $[\mu] \in L_n$ if and only if $\mu \in \text{GL}(n) \cdot [\mu_{\text{he}}]$. Furthermore, $F_n([\mu_{\text{he}}]) = 12$.*

Proof. Assume that $[\mu]$ is a maximum of $F_n : L_n \mapsto \mathbb{R}$. This implies that $[\mu]$ is a critical point of $F_n : \text{GL}(n) \cdot [\mu] \mapsto \mathbb{R}$ and thus $[\mu]$ is a critical point of $F_n : \mathbb{P}V_n \mapsto \mathbb{R}$ (see

Proposition 3.2). But then it follows from Theorem 3.3(i), that $[\mu]$ is also a minimum for $F_n: \mathbf{GL}(n).[\mu] \mapsto \mathbb{R}$, and hence we obtain that $F_n: \mathbf{GL}(n).[\mu] \mapsto \mathbb{R}$ is a constant function. Therefore, every $[\lambda] \in \mathbf{GL}(n).[\mu]$ is a critical point of $F_n: V_n \mapsto \mathbb{R}$ by Proposition 3.2 and so $[\mu]$ must satisfy the following rather strong condition (see Theorem 3.3(ii)):

$$\mathbf{GL}(n).[\mu] = \mathbf{U}(n).[\mu]. \quad (13)$$

In particular, the only possible degeneration of μ is $\mu \mapsto 0$. By Lauret [13, Theorem 5.2] (see also [5]), we have that μ_{he} and μ_{hy} are the only Lie algebras which satisfy (13). It is easy to see that $[\mu_{\text{he}}]$ and $[\mu_{\text{hy}}]$ are critical points of type $(2 < 3 < 4; 2, n-3, 1)$ and $(0 < 1; 1, n-1)$, respectively, and so it follows from Proposition 3.8 that

$$F_n([\mu_{\text{he}}]) = 12 > 4 = F_n([\mu_{\text{hy}}]),$$

concluding the proof. \square

We now prove the main result of this paper, that is, a description of the Lie algebras which are critical points of F_n , in terms of the nilpotent critical points of the functionals F_m with $m \leq n$.

Theorem 4.7. *Let $[\mu] \in L_n$ be a critical point of F_n of type $(0 < k_2 < \dots < k_r; d_1, \dots, d_r)$, and consider $\mathbb{C}^n = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_r$, the eigenspace decomposition of $\mathbf{M}_\mu = c_\mu I + D_\mu$. Then the following conditions hold:*

- (i) $\mathfrak{n} = \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_r$ is the nilradical of μ and $\mu_{\mathfrak{n}} = \mu|_{\mathfrak{n} \times \mathfrak{n}}$ is in his turn a critical point of the functional F_m of type $(k_2 < \dots < k_r; d_2, \dots, d_r)$, where $m = n - d_1$.
- (ii) \mathfrak{g}_1 is a reductive Lie subalgebra of μ .
- (iii) $(\text{ad}_\mu A)^* \in \text{Der}(\mu)$ for any $A \in \mathfrak{g}_1$.

Conversely, let $[\lambda] \in L_m$ be a critical point of F_m of type $(k_2 < \dots < k_r; d_2, \dots, d_r)$, which is necessarily nilpotent, and let $\mathfrak{r} \subset \text{Der}(\lambda)$ be a reductive Lie subalgebra of dimension d_1 such that $A^* \in \text{Der}(\lambda)$ for every $A \in \mathfrak{r}$. Then the semidirect product $\mu = [\cdot, \cdot] \ltimes \lambda$ determines a critical point $[\mu] \in L_n$ of F_n of type $(0 < k_2 < \dots < k_r; d_1, \dots, d_r)$, where $n = d_1 + m$ and $[\cdot, \cdot]$ denotes the Lie bracket of \mathfrak{r} . The fixed inner product considered on $\mathbb{C}^n = \mathfrak{r} \oplus \mathbb{C}^m$ is the extension of the given one on \mathbb{C}^m by setting

$$\langle A, B \rangle = -\frac{4}{c_\lambda} \left(\frac{1}{2} \text{tr} \text{ad} A (\text{ad} B)^* + \text{tr} AB^* \right), \quad A, B \in \mathfrak{r}.$$

Proof. We first prove condition (iii), since it will be crucial in the proof of the other assertions in the theorem. If $X \in \mathfrak{g}_1$ then $[D_\mu, \text{ad}_\mu X] = \text{ad}_\mu D_\mu X = 0$, and

therefore

$$0 = \text{tr}[D_\mu, \text{ad}_\mu X](\text{ad}_\mu X)^* = \text{tr} D_\mu[\text{ad}_\mu X, (\text{ad}_\mu X)^*] = \text{tr} M_\mu[\text{ad}_\mu X, (\text{ad}_\mu X)^*].$$

Thus (iii) follows from Lemma 3.1(iii).

We now prove (ii). Recall that \mathfrak{g}_1 is a subalgebra of μ due to the $\mathbb{Z}_{\geq 0}$ -gradation of μ defined by D_μ . Consider the orthogonal decomposition $\mathfrak{g}_1 = \mathfrak{h} \oplus \mathfrak{a}$, where $\mathfrak{h} = \mu(\mathfrak{g}_1, \mathfrak{g}_1)$. We denote by $[\cdot, \cdot]$ the Lie bracket μ restricted to the subalgebra \mathfrak{g}_1 and by ad its adjoint representation. If $X \in \mathfrak{g}_1$ then by (iii), the maps $\text{ad} X, (\text{ad} X)^*: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ have to leave invariant \mathfrak{h} , which shows that \mathfrak{a} is an abelian factor of \mathfrak{g}_1 and so $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. Since $\text{ad} X$ and $(\text{ad} X)^*$ must also leave invariant the center of \mathfrak{h} for every $X \in \mathfrak{g}_1$, we obtain that the center of \mathfrak{h} equals 0. Moreover, \mathfrak{h} is semisimple. Indeed, since $\text{Ker } B = \{H \in \mathfrak{h} : B(H, \cdot) \equiv 0\}$ (B the Killing form of \mathfrak{h}) is $\text{Der}(\mathfrak{h})$ -invariant, we have that $(\text{Ker } B)^\perp$ is also an ideal of \mathfrak{h} . This implies that $\text{Ker } B$ is a solvable Lie algebra since its Killing form is identically zero. But we have that $[\text{Ker } B, \text{Ker } B] = \text{Ker } B$, and so $\text{Ker } B = 0$, which concludes the proof of (ii).

We finally prove (i). It follows from (ii) that $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the radical of μ . If $A \in \mathfrak{a}$ belongs to the maximal nilpotent ideal of \mathfrak{s} , then $\text{ad}_\mu A : \mathfrak{n} \rightarrow \mathfrak{n}$ is nilpotent. But condition (iii) implies that $(\text{ad}_\mu A)^*$ is a derivation, that is also nilpotent. It is easy to see that this is possible only if $\text{ad}_\mu A = 0$, and this implies that $A = 0$. Indeed, if $\{X_i\}$ is an orthonormal basis of \mathfrak{n} then by (7) we have that

$$\begin{aligned} c_\mu \langle A, A \rangle &= \langle M_\mu A, A \rangle \\ &= -4 \sum_{ij} \langle \mu(A, X_i), X_j \rangle \overline{\langle \mu(A, X_i), X_j \rangle} \\ &\quad + 2 \sum_{ij} \overline{\langle \mu(X_i, X_j), A \rangle} \langle \mu(X_i, X_j), A \rangle \\ &= -4 \text{tr ad}_\mu A (\text{ad}_\mu A)^* = 0. \end{aligned}$$

Thus \mathfrak{n} is the nilradical of μ .

If $H \in \mathfrak{h}$, then by (iii) and the semisimplicity of \mathfrak{h} we have that

$$\text{ad } H = \text{ad}_\mu H|_{\mathfrak{h}} = \frac{1}{2}(\text{ad}_\mu H - (\text{ad}_\mu H)^*)|_{\mathfrak{h}} + \frac{1}{2}(\text{ad}_\mu H + (\text{ad}_\mu H)^*)|_{\mathfrak{h}} = \text{ad } X + \text{ad } Y$$

for some $X, Y \in \mathfrak{h}$, and so $H = X + Y$ since \mathfrak{h} has no center. This implies that there is an orthonormal basis $\{A_i\}$ of \mathfrak{g}_1 such that $\text{ad}_\mu A_i|_{\mathfrak{g}_1}$ is skew-hermitian for any i . By (7) we have that for $X, Y \in \mathfrak{n}$,

$$\begin{aligned} \langle M_\mu X, Y \rangle &= \langle M_{\mu_n} X, Y \rangle - 4 \sum_{ij} \langle \mu(A_i, X), X_j \rangle \overline{\langle \mu(A_i, Y), X_j \rangle} \\ &\quad + 4 \sum_{ij} \overline{\langle \mu(A_i, X_j), X \rangle} \langle \mu(A_i, X_j), Y \rangle \end{aligned}$$

or, in other terms,

$$M_\mu|_{\mathfrak{n}} = M_{\mu_{\mathfrak{n}}} + 4 \sum_i [\operatorname{ad}_\mu A_i, (\operatorname{ad}_\mu A_i)^*]|_{\mathfrak{n}}.$$

Now, by applying Lemma 3.1(ii) for $\mu_{\mathfrak{n}}$ and using (iii), we obtain that

$$0 = \operatorname{tr} M_{\mu_{\mathfrak{n}}} [\operatorname{ad}_\mu A_i, (\operatorname{ad}_\mu A_i)^*]|_{\mathfrak{n}}$$

for all i . In the same way,

$$\begin{aligned} 0 &= \operatorname{tr} M_\mu [\operatorname{ad}_\mu A_i, (\operatorname{ad}_\mu A_i)^*] = \operatorname{tr} M_\mu [\operatorname{ad}_\mu A_i, (\operatorname{ad}_\mu A_i)^*]|_{\mathfrak{n}} \\ &= \operatorname{tr} M_\mu|_{\mathfrak{n}} [\operatorname{ad}_\mu A_i, (\operatorname{ad}_\mu A_i)^*]. \end{aligned}$$

The last two equalities imply that the hermitian operator

$$T = \sum_i [\operatorname{ad}_\mu A_i, (\operatorname{ad}_\mu A_i)^*]|_{\mathfrak{n}}$$

satisfies $\operatorname{tr} T^2 = 0$, and so $T = 0$. Therefore, $M_\mu|_{\mathfrak{n}} = M_{\mu_{\mathfrak{n}}}$, and consequently, $M_{\mu_{\mathfrak{n}}} = c_\mu I + D_\mu|_{\mathfrak{n}}$, that is, $[\mu_{\mathfrak{n}}]$ is a critical point of F_m of type $(k_2 < \dots < k_r, d_2, \dots, d_r)$, as asserted. This concludes the proof of (i).

We now prove the converse assertion. Consider the subalgebra of $\operatorname{Der}(\lambda)$ given by

$$\tilde{\mathfrak{r}} = \{A \in \operatorname{Der}(\lambda) : A^* \in \operatorname{Der}(\lambda)\}.$$

Thus $\tilde{\mathfrak{r}} = \tilde{\mathfrak{k}} + i\tilde{\mathfrak{k}}$, where

$$\tilde{\mathfrak{k}} = \{A \in \operatorname{Der}(\lambda) : A^* = -A\},$$

and so $\tilde{\mathfrak{r}}$ is reductive. In fact, for the inner product on $\tilde{\mathfrak{r}}$ defined by $\langle A, B \rangle_1 = \operatorname{tr} AB^*$, we have that

$$(\operatorname{ad} A)^{*_1} = \operatorname{ad}(A^*) \quad \forall A \in \tilde{\mathfrak{r}}, \quad (14)$$

and hence $\langle \cdot, \cdot \rangle_1$ is $\operatorname{ad} \tilde{\mathfrak{k}}$ -invariant.

The Lie algebra $\tilde{\mathfrak{h}} = [\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$ is semisimple and it is easy to see that each of their simple factors $\tilde{\mathfrak{h}}_i$ satisfies that $A^* \in \tilde{\mathfrak{h}}_i$ for all $A \in \tilde{\mathfrak{h}}_i$. Since $\mathfrak{r} \subset \tilde{\mathfrak{r}}$ and so $\mathfrak{h} = [\mathfrak{r}, \mathfrak{r}] \subset [\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}] = \tilde{\mathfrak{h}}$, we obtain that $A^* \in \mathfrak{h}$ for any $A \in \mathfrak{h}$ as well, and therefore

$$\mathfrak{h} = \mathfrak{k} + i\mathfrak{k}, \quad \text{where } \mathfrak{k} = \{A \in \mathfrak{h} : A^* = -A\}. \quad (15)$$

We define the fixed inner product on $\mathbb{C}^n = \mathfrak{r} \oplus \mathbb{C}^m$, also denoted by $\langle \cdot, \cdot \rangle$, as follows: $\langle \mathfrak{r}, \mathbb{C}^m \rangle = 0$, $\langle \cdot, \cdot \rangle|_{\mathbb{C}^m \times \mathbb{C}^m} = \langle \cdot, \cdot \rangle$ and

$$\langle A, B \rangle = -\frac{4}{c_\lambda} \left(\frac{1}{2} \operatorname{tr} \operatorname{ad} A (\operatorname{ad} B)^{*_1} + \operatorname{tr} AB^* \right), \quad A, B \in \mathfrak{r}. \quad (16)$$

It is easy to see that the adjoints of $\text{ad } B$ with respect to the inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_1$ coincide. It follows from (14) and (15) that there exists an orthonormal basis $\{H_i\}$ of \mathfrak{h} such that $\text{ad}_\mu H_i : \mathbb{C}^n \mapsto \mathbb{C}^n$ is skew hermitian for all i . In particular, there is an orthonormal basis $\{A_i\}$ of \mathfrak{r} such that $\text{ad}_\mu A_i|_{\mathfrak{r}}$ is skew hermitian for all i .

Now, we just have to follow the proof of the first part of the theorem in the converse direction. As in the second part of the proof of (i), we obtain that $M_\mu|_{\mathfrak{n}} = M_\lambda = c_\lambda I + D_\lambda$. If $\{X_i\}$ is an orthonormal basis of \mathbb{C}^m then for $A \in \mathfrak{r}$ and $X \in \mathbb{C}^m$ we have that

$$\langle M_\mu X, A \rangle = -4 \sum_{ij} \langle \mu(X, X_i), X_j \rangle \overline{\langle \mu(A, X_i), X_j \rangle} = -4 \text{tr ad}_\lambda X A^* = 0,$$

since A^* is the sum of a skew-hermitian and a hermitian derivation of λ and $\text{ad}_\lambda X$ is nilpotent. If $A, B \in \mathfrak{r}$ and we now denote by $\{X_i\}$ an orthonormal basis of \mathbb{C}^n which contains $\{H_i\}$, then by (7) we have that

$$\begin{aligned} \langle M_\mu A, B \rangle &= -4 \sum_{ij} \langle \mu(A, X_i), X_j \rangle \overline{\langle \mu(B, X_i), X_j \rangle} \\ &\quad + 2 \sum_{ij} \overline{\langle \mu(X_i, X_j), A \rangle} \langle \mu(X_i, X_j), B \rangle \\ &= -4 \text{tr ad}_\mu A (\text{ad}_\mu B)^* + 2 \sum_{ij} \overline{\langle \mu(H_i, H_j), A \rangle} \langle \mu(H_i, H_j), B \rangle \\ &= -2 \text{tr ad } A (\text{ad } B)^{*1} - 4 \text{tr } AB^*. \end{aligned} \quad (17)$$

It then follows easily from (15), (17) and the definition of $\langle \cdot, \cdot \rangle|_{\mathfrak{r} \times \mathfrak{r}}$ given in (16) that $M_\mu|_{\mathfrak{r}} = c_\lambda I$. We conclude that $M_\mu = c_\lambda I + \tilde{D}_\lambda$, where $\tilde{D}_\lambda|_{\mathfrak{r}} \equiv 0$ and $\tilde{D}_\lambda|_{\mathbb{C}^m} = D_\lambda$, and hence μ is a critical point of type $(0 < k_2 < \dots < k_r; d_1, \dots, d_r)$, as it was to be shown. \square

From the proof of the theorem, we can deduce the following compatibility properties at a critical point $[\mu]$ of F_n , between the Lie bracket μ and the fixed inner product on \mathbb{C}^n .

Proposition 4.8. *Let $[\mu] \in L_n$ be a critical point of F_n of type $(0 < k_2 < \dots < k_r; d_1, \dots, d_r)$ (i.e. $k_1 = 0$), and consider $\mathbb{C}^n = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_r$, the eigenspace decomposition of $M_\mu = c_\mu I + D_\mu$. Let $\mathfrak{g}_1 = \mathfrak{h} \oplus \mathfrak{a}$ with \mathfrak{h} semisimple and \mathfrak{a} the center of \mathfrak{g}_1 . Then the following conditions hold:*

- (i) $\text{ad}_\mu A$ is a normal operator for every $A \in \mathfrak{a}$.
- (ii) The real subalgebra $\mathfrak{k} = \{A \in \mathfrak{h} : (\text{ad}_\mu A)^* = -\text{ad}_\mu A\}$ is a maximal compact subalgebra of \mathfrak{h} , that is, $\mathfrak{h} = \mathfrak{k} + \mathfrak{i}\mathfrak{k}$.
- (iii) The hermitian inner product on \mathfrak{g}_1 satisfies

$$\langle A, B \rangle = -\frac{4}{c_\mu} \left(\frac{1}{2} \text{tr ad}_\mu A (\text{ad}_\mu B)^*|_{\mathfrak{h}} + \text{tr ad}_\mu A (\text{ad}_\mu B)^*|_{\mathfrak{n}} \right), \quad A, B \in \mathfrak{g}_1.$$

Proof. Parts (ii) and (iii) follow directly from the proof of Theorem 4.7. We now prove (i). If $A \in \mathfrak{a}$ then $(\operatorname{ad}_\mu A)^* \in \operatorname{Der}(\mu)$ by Theorem 4.7(iii). This implies that

$$0 = \operatorname{ad}_\mu((\operatorname{ad}_\mu A)^* A) = [(\operatorname{ad}_\mu A)^*, \operatorname{ad}_\mu A],$$

which concludes the proof. \square

Roughly speaking, Theorem 4.7 says that the study of the Lie algebras which are critical points of F_n reduces to the understanding of those which are nilpotent. In other words, it suffices to describe the Kirwan–Ness quotient $\mathcal{N}_n // GL(n)$, where $\mathcal{N}_n \subset \mathcal{L}_n \subset \mathbb{P}V_n$ is the algebraic subvariety of all nilpotent Lie algebras.

Remark 4.9. We would like to point out a somewhat mysterious characterization of the Kirwan–Ness quotient $\mathcal{N}_n // GL(n)$ (over \mathbb{R}) in terms of Riemannian geometry: the classification of the nilpotent critical points of $F_n: L_n \mapsto \mathbb{R}$ is equivalent to the classification up to isometry of all left invariant Riemannian metrics on nilpotent Lie groups which are Ricci solitons (see [11]), and also of all Einstein left invariant Riemannian metrics on solvable Lie groups (see [6,12]). A crucial point here is that the moment map M_μ coincides with the Ricci operator of a certain Riemannian metric naturally associated with μ .

We now give some known examples of nilpotent critical points, most of which come from the interplay mentioned above.

Example 4.10. (i) A two-step nilpotent Lie algebra μ is a critical point of F_n of type $(1 < 2; d_1, d_2)$ if and only if $M_\mu|_{\mathfrak{z}(\mu)}$ and $M_\mu|_{\mathfrak{z}(\mu)^\perp}$ are both a multiple of the identity, where $\mathfrak{z}(\mu)$ is the center of μ and $\dim \mathfrak{z}(\mu) = d_2$. Thus any Heisenberg-type Lie algebra (see [1]) and any two-step nilpotent Lie algebra constructed via a representation of a compact Lie group (see [10]), contain a critical point in their orbits (over \mathbb{R}). In [4], curves of critical points of type $(1 < 2; 5, 5)$ and $(1 < 2; 6, 3)$ are given.

(ii) Any nilpotent Lie algebra with a codimension one abelian ideal contains a critical point of F_n in its orbit (see Section 5).

(iii) Every nilpotent Lie algebra of dimension ≤ 5 is isomorphic to a critical point of F_n (see [14]).

(iv) The lowest possible dimension for the existence of a curve of nonisomorphic nilpotent critical points of F_n is $n = 7$, and an example of such a curve is given in [14].

(v) It has been recently proved in [19] that any six-dimensional nilpotent Lie algebra contains a critical point of F_6 in its orbit.

On the other hand, since any nilpotent critical point is necessarily \mathbb{N} -graded, we have that a characteristically nilpotent Lie algebra (i.e. $\operatorname{Der}(\mu)$ nilpotent)

can never be a critical point of F_n . The first dimension where these Lie algebras appear is $n = 7$.

5. Lie algebras with a codimension one abelian ideal

We consider the closed subset of \mathcal{L}_{n+1} given by

$$\mathcal{A} = \{\mu \in \mathcal{L}_{n+1} : \mu \text{ has an abelian ideal of dimension } n\}.$$

The aim of this section is to illustrate, via a study of \mathcal{A} , most of the notions considered in this paper. It follows from [13, Theorem 6.2] that if a Lie algebra degenerates to an element in \mathcal{A} , then it has to lie in \mathcal{A} . This implies that the stratum $S_{(0 < 1; 1, n)} \subset \mathcal{A}$ (see (12)).

By fixing a decomposition $\mathbb{C}^{n+1} = \mathbb{C}H \oplus \mathbb{C}^n$, every element in \mathcal{A} is isomorphic to a $\mu \in \mathcal{A}$ having \mathbb{C}^n as the required abelian ideal. In this case, μ is determined by the matrix $A = \text{ad}_\mu H|_{\mathbb{C}^n} \in \mathfrak{gl}(n)$, and so it will be denoted by μ_A . It is easy to see that μ_A is isomorphic to μ_B , $A, B \in \mathfrak{gl}(n)$, if and only if A is conjugate to B up to scaling. In other words, isomorphism classes of \mathcal{A} are parameterized by $\mathfrak{gl}(n)/GL(n)$, under the action

$$\varphi.A = (\det \varphi)\varphi A \varphi^{-1}, \quad \varphi \in GL(n), \quad A \in \mathfrak{gl}(n). \quad (18)$$

It is not difficult to prove that for a nonnilpotent A ,

$$\text{Der}(\mu_A) = \left\{ \begin{bmatrix} 0 & 0 \\ * & B \end{bmatrix} : BA = AB, B \in \mathfrak{gl}(n) \right\}, \quad (19)$$

and if A is nilpotent then

$$\text{Der}(\mu_A) = \mathbb{C}D \oplus \left\{ \begin{bmatrix} 0 & 0 \\ * & B \end{bmatrix} : BA = AB, B \in \mathfrak{gl}(n) \right\}, \quad (20)$$

where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & D_1 \end{bmatrix}, \quad [D_1, A] = A.$$

Using formula (7) as in the proof of Theorem 4.7, we can calculate the moment map and the functional F_{n+1} on \mathcal{A} . We assume that the decomposition $\mathbb{C}^{n+1} = \mathbb{C}H \oplus \mathbb{C}^n$ is orthogonal and $\|H\| = 1$.

Proposition 5.1. *For any $A \in \mathfrak{gl}(n)$ we have that*

$$M_{\mu_A} = \begin{bmatrix} -4 \operatorname{tr} AA^* & 0 \\ 0 & 4[A, A^*] \end{bmatrix}.$$

Consequently, $\|\mu_A\|^2 = 2 \operatorname{tr} AA^*$ and $F_{n+1}([\mu_A]) = 4 + 16 \operatorname{tr} [A, A^*]^2$.

We deduce from the formula of $F_{n+1}([\mu_A])$ that F_{n+1} measures how far is A from being normal, and so if A is semisimple then the orbit $GL(n+1) \cdot [\mu_A]$ will contain a critical point of F_{n+1} (see Theorem 3.3). Indeed, for any semisimple A there exists $\varphi \in GL(n)$ such that $\varphi A \varphi^{-1}$ is normal with respect to $\langle \cdot, \cdot \rangle$, and hence if $\psi \in GL(n+1)$ is defined by $\psi|_{\mathbb{C}H} = 1$ and $\psi|_{\mathbb{C}^n} = \varphi$, then $\psi \cdot \mu_A = \mu_{\varphi A \varphi^{-1}}$, and so $\psi \cdot \mu_A$ is a critical point of type $(0 < 1; 1, n)$. However as a counterpart, there is also a critical point of F_{n+1} in the orbits of the nilpotent μ_A 's.

Proposition 5.2. *The orbit $GL(n+1) \cdot [\mu_A]$ contains a critical point of F_{n+1} if and only if A is semisimple or nilpotent. If A is not nilpotent, then $[\mu_A]$ is a critical point of F_{n+1} if and only if A is normal, and in this case, it is of type $(0 < 1; 1, n)$.*

Proof. We first assume that A is not nilpotent. If $[\mu_A]$ is a critical point of F_{n+1} then $M_{\mu_A} \in \mathbb{R}I \oplus \operatorname{Der}(\mu_A)$ (see Proposition 3.2(iii)). Thus it follows from (19) and Proposition 5.1 that $[A, A^*]$ commutes with A , and hence

$$\begin{bmatrix} 0 & 0 \\ 0 & [A, A^*] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & A^* \end{bmatrix} \end{bmatrix}$$

is a derivation of μ_A . Now, parts (ii) and (iii) of Lemma 3.1 imply that

$$\begin{bmatrix} 0 & 0 \\ 0 & A^* \end{bmatrix}$$

is also a derivation of μ_A , concluding that A is normal (see (19)). The type of $[\mu_A]$ can be deduced from the formula

$$M_{\mu_A} = -4 \operatorname{tr} AA^* I + \begin{bmatrix} 0 & 0 \\ 0 & 4 \operatorname{tr} AA^* I \end{bmatrix}.$$

We now consider A nilpotent. For each r -tuple (n_1, \dots, n_r) of integer numbers satisfying $n_1 \geq \dots \geq n_r \geq 0$ and $(n_1 + 1) + \dots + (n_r + 1) = n$, we consider the $n \times n$ -matrix $A_{(n_1, \dots, n_r)}$ obtained by the direct sum of the r blocks of

the form

$$\begin{bmatrix} 0 & & & & & \\ & \frac{1}{n_i^2} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & (jn_i - j(j-1))^{\frac{1}{2}} & 0 \\ & & & & & \ddots & \\ & & & & & & \ddots & \\ & & & & & & & \frac{1}{n_i^2} & 0 \end{bmatrix}, \quad \text{where } j = 1, \dots, n_i.$$

By a straightforward computation, one can see that

$$M_{\mu_{A(n_1, \dots, n_r)}} \in \mathbb{R}I \oplus \text{Der}(\mu_{A(n_1, \dots, n_r)}),$$

and hence $[\mu_{A(n_1, \dots, n_r)}]$ is a critical point of F_{n+1} for any (n_1, \dots, n_r) . If $A \in \mathfrak{gl}(n)$ is nilpotent then μ_A is isomorphic to $[\mu_{A(n_1, \dots, n_r)}]$, where $n_1 + 1 \geq \dots \geq n_r + 1$ are the dimensions of the Jordan blocks of A . Thus every orbit $GL(n+1) \cdot [\mu_A]$ with A nilpotent contains a critical point of F_{n+1} . \square

Therefore, the Kirwan–Ness quotient is given by

$$\mathcal{A} // GL(n+1) = \{\mu_A : A \text{ semisimple or nilpotent}\} = \mathbb{P}\mathbb{C}^n / S_n \cup \bigcup \{\mu_{A(n_1, \dots, n_r)}\}$$

for $n_1 \geq \dots \geq n_r \geq 0$, $n_1 + \dots + n_r = n - r$ (see the proof of the above proposition), where S_n denotes the symmetric group permuting the coordinates of $\mathbb{P}\mathbb{C}^n$.

Consider for each $A \in \mathfrak{gl}(n)$ the decomposition $A = S + N$ in their semisimple and nilpotent parts. If $S \neq 0$, it is easy to see that the $-\text{grad}(F_{n+1})$ -flow starting from the point $[\mu_A]$ converges to the critical point $[\mu_S]$. Thus the stratum of type $(0 < 1; 1, n)$ is given by

$$S_{(0 < 1; 1, n)} = \bigcup_{S \neq 0} GL(n+1) \cdot \mu_A = \{\mu \in \mathcal{A} : \mu \text{ is not nilpotent}\}.$$

It is proved in [14] that the type of the critical point $[\mu_{A(n_1, \dots, n_r)}]$ is given by

- $(1 < \theta - \frac{n_1}{2} < \theta - \frac{n_1}{2} + 1 < \dots < \dots < \theta + \frac{n_1}{2}; 1, \dots)$, if $n_i \equiv \varepsilon(2)$, for each i ,
- $(2 < 3 < 4; 2, n-3, 1)$, if $n_1 = 1$, $n_i = 0$ for each $i \geq 2$,
- $(2 < 2\theta - n_e < 2\theta - n_e + 2 < \dots < 2\theta + n_o; 1, \dots)$, otherwise,

where n_e and n_o are the greatest even and odd numbers among the n_i 's respectively and

$$\theta = 1 + \frac{n_1(n_1+1)(n_1+2) + \dots + n_r(n_r+1)(n_r+2)}{12}.$$

It is easy to check that the type of $\left[\mu_{A(n_1, \dots, n_r)}\right]$ coincides with the type of $\left[\mu_{A(n'_1, \dots, n'_r)}\right]$ if and only if $n_i = n'_i$ for all i . This implies that the other strata $S_\alpha \cap \mathcal{A}$ are just the orbits $GL(n+1) \cdot [\mu_A]$ for each nilpotent A .

6. Critical points and closed orbits

In this section, we show how the moduli space of isomorphism classes of critical points of F_n of a given type can be viewed as the categorical quotient of a suitable action (see [17, Section 9] for the general case). We fix a type $\alpha = (k_1 < \dots < k_r; d_1, \dots, d_r)$, and denote by

$$V_\alpha = \{\mu \in V_n : D_\alpha \in \text{Der}(\mu)\}, \quad G_\alpha = Z_{GL(n)}(D_\alpha) = GL(d_1) \times \dots \times GL(d_r),$$

where

$$D_\alpha = \begin{bmatrix} k_1 I_{d_1} & & \\ & \ddots & \\ & & k_r I_{d_r} \end{bmatrix},$$

and I_{d_i} denotes the $d_i \times d_i$ identity matrix. We consider the reductive subgroup of G_α defined by

$$\tilde{G}_\alpha = \left\{ g \in G_\alpha : \prod_{i=1}^r (\det g_i)^{k_i} = \det g = 1 \right\}.$$

We are interested in the action of \tilde{G}_α on V_α . The corresponding Lie algebras satisfy

$$\mathfrak{g}_\alpha = \tilde{\mathfrak{g}}_\alpha \oplus \mathbb{C}I \oplus \mathbb{C}D_\alpha, \quad \tilde{\mathfrak{g}}_\alpha = \{A \in \mathfrak{g}_\alpha : \text{tr } AD_\alpha = \text{tr } A = 0\}.$$

Let $\mathfrak{k}_\alpha, \tilde{\mathfrak{k}}_\alpha$ denote the Lie algebras of the maximal compact subgroups. It is proved in [17] that the moment map $m : \mathbb{P}V_\alpha \mapsto \mathfrak{i}\mathfrak{k}_\alpha$ corresponding to the action of G_α on V_α is just the restriction of $m : \mathbb{P}V \mapsto \mathfrak{i}\mathfrak{u}(n)$ to $\mathbb{P}V_\alpha$, and the moment map $\tilde{m} : \mathbb{P}V_\alpha \mapsto \mathfrak{i}\tilde{\mathfrak{k}}_\alpha$ of the \tilde{G}_α -action on V_α coincides with the composition of $m : \mathbb{P}V_\alpha \mapsto \mathfrak{i}\mathfrak{k}_\alpha$ with the orthogonal projection $\mathfrak{i}\mathfrak{k}_\alpha \mapsto \mathfrak{i}\tilde{\mathfrak{k}}_\alpha$.

Recall that $[\mu] \in \mathbb{P}V_\alpha$ is a critical point of type α if and only if $m(\mu) = M_\mu \in \mathbb{C}(c_\alpha I + D_\alpha)$, which is equivalent to $M_\mu \in \mathbb{C}I \oplus \mathbb{C}D_\alpha$. Since $\mathfrak{i}\mathfrak{k}_\alpha = \mathfrak{i}\tilde{\mathfrak{k}}_\alpha \oplus (\mathbb{C}I + \mathbb{C}D_\alpha)$ is an orthogonal decomposition, it follows that

$$\tilde{m}^{-1}(0) = \mathbb{P}V_\alpha \cap C_\alpha,$$

where C_α is the set of critical points of $F_n : \mathbb{P}V_n \mapsto \mathbb{R}$ of type α . This means that the orbit $\tilde{G}_\alpha \cdot \mu$ of $\mu \in V_\alpha$ is closed if and only if $\tilde{G}_\alpha \cdot [\mu]$ intersects C_α . The open set of semistable points is precisely

$$\mathbb{P}V_\alpha^{\text{ss}} = \mathbb{P}V_\alpha \cap S_\alpha, \quad (21)$$

and the categorical quotient $\mathbb{P}V_\alpha/\tilde{G}_\alpha$ coincides with

$$\mathbb{P}V_\alpha \cap C_\alpha/\tilde{K}_\alpha = C_\alpha/U(n),$$

which parameterizes the set of critical points of type α up to isomorphism. This shows that $C_\alpha/U(n)$ is a projective algebraic variety. If $\alpha_1, \dots, \alpha_s$ denote the different types of critical points of $F_n: L_n \mapsto \mathbb{R}$ then the stratification of the Kirwan–Ness quotient described in Section 2 is given by

$$L_n//GL(n) = X_{\alpha_1} \cup \dots \cup X_{\alpha_s} \quad (\text{disjoint union}),$$

where each X_{α_i} is homeomorphic to $(L_n \cap C_{\alpha_i})/U(n)$ and the following frontier property holds:

$$\bar{X}_\alpha \subset X_\alpha \cup \bigcup_{\beta > \alpha} X_\beta.$$

Recall that $\alpha < \beta$ if and only if the corresponding critical values satisfy $F_n(\alpha) < F_n(\beta)$ (see Proposition 3.8).

Example 6.1. Consider the type $\alpha = (0; n)$. We have that $V_\alpha = V_n$, $G_\alpha = GL(n)$ and $\tilde{G}_\alpha = SL(n)$. Thus, the set of semistable points $\mathbb{P}V_n^{\text{ss}} = S_\alpha$ is open in $\mathbb{P}V_n$ and consequently $S_\alpha \cap L_n$ is open in L_n . We know that $S_\alpha \cap L_n$ is precisely the set of n -dimensional semisimple Lie algebras (see Remark 4.4). If $S_\alpha \cap L_n = GL(n) \cdot [\mu_1] \cup \dots \cup GL(n) \cdot [\mu_m]$, where μ_1, \dots, μ_m are the n -dimensional semisimple Lie algebras up to isomorphism, then it follows from the fact that $\dim \text{Der}(\mu_j) = n$ for all j that $\overline{GL(n) \cdot [\mu_j]} \cap S_\alpha \cap L_n = GL(n) \cdot [\mu_j]$, and so $GL(n) \cdot [\mu_j]$ is closed in $S_\alpha \cap L_n$ for any j . This implies that $GL(n) \cdot [\mu_i]$ is open in $S_\alpha \cap L_n$ and so in L_n for any i . As an immediate consequence, $GL(n) \cdot \mu_i$ is open in \mathcal{L}_n for any i . We therefore obtain an alternative proof of the rigidity of semisimple Lie algebras, which makes use of any cohomology argument.

On the other hand, we also conclude from Remark 4.4 that an orbit $SL(n) \cdot \mu \subset \mathcal{L}_n$ is closed if and only if μ is semisimple, and any non-semisimple Lie algebra λ is in the null-cone. Thus, the categorical quotient is $L_n/SL(n) = \{\mu_1, \dots, \mu_m\}$.

We now summarize the results obtained in the above example, as a corollary of Theorem 4.3.

Corollary 6.2. (i) *The $GL(n)$ -orbit of any semisimple Lie algebra is open in \mathcal{L}_n .*

(ii) *The $SL(n)$ -orbit of $\mu \in \mathcal{L}_n$ is closed if and only if μ is semisimple.*

(iii) *$0 \in \overline{SL(n) \cdot \lambda}$ for any nonsemisimple $\lambda \in \mathcal{L}_n$.*

Example 6.3. If $\alpha = (0 < 1; 1, n)$, then $V_\alpha = \{\mu_A : A \in \mathfrak{gl}(n)\} \subset \mathcal{A} \subset \mathcal{L}_{n+1}$ (see Section 5), $G_\alpha = \mathbb{C}^* \times GL(n)$ and $\tilde{G}_\alpha = \{1\} \times SL(n)$. The semistable points are

$$V_\alpha^{\text{ss}} = V_\alpha \cap S_\alpha = \{\mu_A : A \text{ is not nilpotent}\}$$

and the null-cone $N = \{\mu_A : A \text{ is nilpotent}\}$. An orbit $\tilde{G}_\alpha \cdot \mu_A$ is closed if and only if A is semisimple, and the categorical quotient is given by

$$\mathbb{P}V_\alpha / \tilde{G}_\alpha = \mathbb{P}\mathbb{C}^n / S_n,$$

which parameterizes diagonal matrices up to conjugation and scaling.

In what follows, we completely develop the case $n = 4$. A list of all four-dimensional complex Lie algebras up to isomorphism is given in Table 1 (see [2]). In Table 2 we give the stratification for L_4 . For each type α , we describe the stratum S_α and the categorical quotient $(L_4 \cap \mathbb{P}V_\alpha) / \tilde{G}_\alpha = (L_4 \cap C_\alpha) / \mathrm{U}(4)$, as well as the critical values. We also denote by \mathfrak{g} the $GL(n)$ -orbit in L_4 of Lie algebras isomorphic to \mathfrak{g} .

For $\alpha = (0 < 1 < 2; 1, 2, 1)$, we have that $S_\alpha = \{\mathfrak{g}_8(c) : c \in \mathbb{C}\} \cup \{\mathfrak{g}_6, \mathfrak{g}_7\}$. All of these orbits meet C_α excepting $\mathfrak{g}_8(\frac{1}{4})$, for which the negative gradient flow of F_4 converges to the critical point in \mathfrak{g}_6 . Recall that the case $\alpha = (0 < 1; 1, 3)$ has been studied in Section 5. Thus the Kirwan–Ness quotient $L_4 // GL(4)$ classifies all four-dimensional Lie algebras except for $\mathfrak{g}_8(\frac{1}{4})$, $\mathfrak{g}_3(\frac{27}{4})$, \mathfrak{g}_5 , $\mathfrak{g}_2(\frac{1}{27}, \frac{1}{3})$ and $\mathfrak{g}_2(\frac{\gamma}{(\gamma+2)^3}, \frac{2\gamma+1}{(\gamma+2)^2})$, $\gamma \in \mathbb{C} \setminus \{2\}$. The negative gradient flow of F_4 starting from $\mathfrak{g}_3(\frac{27}{4})$, \mathfrak{g}_5 , $\mathfrak{g}_2(\frac{1}{27}, \frac{1}{3})$ and $\mathfrak{g}_2(\frac{\gamma}{(\gamma+2)^3}, \frac{2\gamma+1}{(\gamma+2)^2})$ converges to the critical point in $\mathfrak{g}_1(-2)$, $\mathfrak{g}_1(1)$, $\mathfrak{g}_1(1)$ and $\mathfrak{g}_1(\gamma)$, respectively.

The classification of all possible degenerations for four-dimensional complex Lie algebras given in [2] can be used to check the validity of the frontier property of the stratifications of L_4 and $L_4 // GL(n)$.

We note that the closure $\overline{S_\alpha}$ of any nonnilpotent stratum S_α (i.e. $k_1 = 0$) gives rise to an irreducible component of \mathcal{L}_4 . One can see that this is true for every \mathcal{L}_n with $n \leq 7$, by using the results given in [3]. We do not know if the irreducible components of \mathcal{L}_n can be described in this way for any value of n .

Example 6.4. The approach considered in this section has been used successfully by Galitski and Timashev [4] in the study of two-step nilpotent Lie algebras. For a type of the form $\alpha = (1 < 2; d_1, d_2)$ one gets $V_\alpha = \Lambda^2(\mathbb{C}^{d_1})^* \otimes \mathbb{C}^{d_2}$ and $\tilde{G}_\alpha = SL(d_1) \times SL(d_2)$. Thus, the closed \tilde{G}_α -orbits in V_α are precisely those which contain a critical point of type α . In [4] the quotient space $V_\alpha / \tilde{G}_\alpha$ is studied by using methods in geometric invariant theory, in the cases $(d_1, d_2) = (5, 5)$ and $(6, 3)$. This allows to complete the classification of two-step nilpotent Lie algebras of dimension ≤ 9 . They exploit the fact that in these cases, the action of \tilde{G}_α on V_α is not only visible (i.e. $p^{-1}(p(\mu))$ contains finitely many orbits for any μ , where $p : V_\alpha \mapsto V_\alpha / \tilde{G}_\alpha$), but it is also a θ -group.

Remark 6.5. As a consequence of Theorem 4.7, we have that any nilpotent critical point of F_n admits an \mathbb{N} -gradation. We do not know if the converse assertion is true. For instance, if the representations $\Lambda^2 GL(d_1)^* \otimes GL(d_2)$ with $d_1 +$

Table 1
Complex Lie algebras of dimension 4

\mathfrak{g}	Lie brackets
\mathbb{C}^4	—
$\mathfrak{n}_3 \oplus \mathbb{C}$	$[x_1, x_2] = x_3$
$\mathfrak{r}_2 \oplus \mathbb{C}^2$	$[x_1, x_2] = x_1$
$\mathfrak{r}_3 \oplus \mathbb{C}$	$[x_1, x_2] = x_2, [x_1, x_3] = x_2 + x_3$
$\mathfrak{r}_{3,\lambda} \oplus \mathbb{C}$	$[x_1, x_2] = x_2, [x_1, x_3] = \lambda x_3, \lambda \in \mathbb{C}, 0 < \lambda \leq 1$
$\mathfrak{r}_2 \oplus \mathfrak{r}_2$	$[x_1, x_2] = x_1, [x_3, x_4] = x_3$
$\mathfrak{sl}_2 \oplus \mathbb{C}$	$[x_1, x_2] = x_3, [x_1, x_3] = -2x_1, [x_2, x_3] = 2x_2$
\mathfrak{n}_4	$[x_1, x_2] = x_3, [x_1, x_3] = x_4$
$\mathfrak{g}_1(\alpha)$	$[x_1, x_2] = x_2, [x_1, x_3] = x_3, [x_1, x_4] = \alpha x_4, \alpha \in \mathbb{C}^*$
$\mathfrak{g}_2(\alpha, \beta)$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = \alpha x_2 - \beta x_3 + x_4,$ $\alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \text{ or } \alpha, \beta = 0$
$\mathfrak{g}_3(\alpha)$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = \alpha(x_2 + x_3), \alpha \in \mathbb{C}^*$
\mathfrak{g}_4	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_2$
\mathfrak{g}_5	$[x_1, x_2] = \frac{1}{3}x_2 + x_3, [x_1, x_3] = \frac{1}{3}x_3, [x_1, x_4] = \frac{1}{3}x_4$
\mathfrak{g}_6	$[x_1, x_2] = x_2, [x_1, x_3] = x_3, [x_1, x_4] = 2x_4, [x_2, x_3] = x_4$
\mathfrak{g}_7	$[x_1, x_2] = x_3, [x_1, x_3] = x_2, [x_2, x_3] = x_4$
$\mathfrak{g}_8(\alpha)$	$[x_1, x_2] = x_3, [x_1, x_3] = -\alpha x_2 + x_3, [x_1, x_4] = x_4, [x_2, x_3] = x_4, \alpha \in \mathbb{C}$

Table 2
Stratification for L_4

α (type)	S_α	$C_\alpha/\mathbf{U}(4)$	$F_4(C_\alpha)$
$(0 < 1; 3, 1)$	$\mathfrak{sl}_2 \oplus \mathbb{C}$	$\{p\}$	$\frac{4}{3}$
$(0 < 1; 2, 2)$	$\mathfrak{r}_2 \oplus \mathfrak{r}_2$	$\{p\}$	2
$(0 < 1 < 2; 1, 2, 1)$	Solvable with nilradical \mathfrak{n}_3	$\mathbb{C} \cup \{p\}$	3
$(0 < 1; 1, 3)$	Solvable with nilradical \mathbb{C}^3	$\mathbb{P}\mathbb{C}^3/S^3$	4
$(1 < 2 < 3 < 4; 1, 1, 1, 1)$	\mathfrak{n}_4	$\{p\}$	6
$(2 < 3 < 4; 2, 1, 1)$	$\mathfrak{n}_3 \oplus \mathbb{C}$	$\{p\}$	12

$d_2 = n$ were nice enough, in the sense that any orbit contains a critical point of F_n , then we would be able to describe the moduli space of all n -dimensional two-step nilpotent Lie algebras up to isomorphism as a finite union of categorical quotients. However, we have reasons to believe that this would be too optimistic for large values of n .

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